# Linear Algebra Done Right Chapter 1

## Example 1.4 Show that $ab = ba$ for all $a, b \in \mathbf{C}$

Let $a, b \in \mathbf{C}$. Then ~~[illegible]~~ $a = x + yi$ and $b = z + wi$ for some $x, y, z, w \in \mathbf{R}$. So $ab = (x+yi)(z+wi) = (xz - yw) + (xw + yz)i = (zx - wy) + (zy + wx)i = (z+wi)(x+yi) = ba$. □

## 1.A.1 Suppose $a$ and $b$ are real numbers, not both 0. Find real numbers $c$ and $d$ such that $1/(a+bi) = c + di$.

Then $1 = (a+bi)(c+di) = (ac - bd) + (ad + bc)i$. Since $1 = 1 + 0i = (ac-bd) + (ad+bc)i$, we know that $ac - bd = 1$ and $ad + bc = 0$. Solve both equations for $c$ to get $\frac{bd+1}{a} = -\frac{ad}{b}$. Multiply both sides by $ab$ to get $b(bd+1) = -a^2d$. So $b^2d + b = -a^2d$. Add $a^2d - b$ to both sides to get $-b = b^2d + a^2d = d(a^2+b^2)$. Finally, divide both sides by $a^2+b^2$ to see that $d = -\frac{b}{a^2+b^2}$.

We revisit the equations $ac - bd = 1$ and $ad + bc = 0$ and this time solve both for $d$ to get $\frac{ac-1}{b} = -\frac{bc}{a}$. Multiply both ~~sides by ab to get~~ sides by $ab$ to get $-b^2c = a(ac-1) = a^2c - a$. Add $+b^2c + a$ to both sides to obtain $a = a^2c + b^2c = (a^2+b^2)c$. To finish, divide both sides by $a^2 + b^2$ to see that $c = \frac{a}{a^2+b^2}$.

So $c = a/(a^2+b^2)$ and $d = -b/(a^2+b^2)$. □

## 1.A.2 Show that $(-1+\sqrt{3}i)/2$ is a cube root of 1.

$$\frac{-1+\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2} = \frac{-2-2\sqrt{3}i}{4} \cdot \frac{-1+\sqrt{3}i}{2} = \frac{8+0i}{8} = 1 \quad \square$$

## 1.A.3 Find two distinct square roots of $i$.

$\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}}i$ and $-\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2}}i$

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**1.A.6 Show that $(ab)c = a(bc)$ for all $a,b,c \in \mathbf{C}$.**

Then $a = x+yi$, $b = z+wi$, and $c = r+si$ for some $x,y,z,w,r,s \in \mathbf{R}$.
So $(ab)c = ((x+yi)(z+wi))c = ((xz-yw)+(xw+yz)i)(r+si) =$
$((xz-yw)r-(xw+yz)s)+((xz-yw)s+(xw+yz)r)i =$
$(xzr-ywr-xws-yzs)+(xzs+xwr+yzr-yws)i =$
$((zr-ws)x-(wr+zs)y)+((zr-ws)y+(wr+zs)x)i =$
$(x+yi)((zr-ws)+(wr+zs)i) = a((z+wi)(r+si)) = a(bc)$. $\square$

**1.A.7 Show that for every $a \in \mathbf{C}$, there exists a unique $b \in \mathbf{C}$ such that $a+b=0$.**

Let $x+yi \in \mathbf{C}$. Then $x,y \in \mathbf{R}$. Observe $(x+yi)+(-x-yi) = (x-x)+(y-y)i =$
$0+0i = 0$. Since $-x-yi \in \mathbf{C}$, we have the desired result. $\square$

**1.A.8 Show that for every $a \in \mathbf{C}$ with $a \neq 0$, there exists a unique $b \in \mathbf{C}$ such that $ab = 1$.**

Let $(x+yi) \in \mathbf{C}$. Then $x,y \in \mathbf{R}$. Observe:
$$(x+yi)\left(\frac{x}{x^2+y^2}-\left(\frac{y}{x^2+y^2}\right)i\right) = \left(\left(\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}\right)+\left(\frac{xy}{x^2+y^2}-\frac{xy}{x^2+y^2}\right)i\right) =$$
$$\left(\frac{x^2+y^2}{x^2+y^2}\right)+\left(\frac{0}{x^2+y^2}\right)i = 1+0i = 1$$
Since $\frac{x}{x^2+y^2}-\left(\frac{y}{x^2+y^2}\right)i$ is a complex number, we have the desired result. $\square$

**1.A.9 Show that $c(a+b) = ca+cb$ for all $a,b,c \in \mathbf{C}$.**

Let $a+bi,\ c+di,\ x+yi \in \mathbf{C}$. We must show that
$(x+yi)((a+bi)+(c+di)) = (x+yi)(a+bi)+(x+yi)(c+di)$. Observe
$(x+yi)((a+bi)+(c+di)) =$
$(x+yi)((a+c)+(b+d)i) = (x(a+c)-y(b+d))+(x(b+d)+y(a+c)i) =$
$(xa+xc-yb-yd)+(xb+xd+ya+yc)i =$
$((xa-yb)+(xc-yd))+((xb+ya)+(xd+yc))i =$
$((xa-yb)+(xb+ya)i)+((xc-yd)+(xd+yc)i) =$
$(x+yi)(a+bi)+(x+yi)(c+di)$. $\square$

## 1.A.10 Find $x \in \mathbf{R}^4$ such that $(4,-3,1,7)+2x=(5,9,-6,8)$.

$x=(\frac{1}{2},6,-\frac{7}{2},\frac{1}{2})$. Observe $(4,-3,1,7)+2(\frac{1}{2},6,-\frac{7}{2},\frac{1}{2})=(4,-3,1,7)+(1,12,-7,1)=(5,9,-6,8)$.

## 1.A.11 Explain why there does not exist $x \in \mathbf{C}$ such that $x(2-3i,5+4i,-6+7i)=(12-5i,7+22i,-32-9i)$.

~~Assume, for the purpose of contradicting, that such a complex number x exists. Then $(12-5i,7+22i,-32-9i)=x(2-3i,5+4i,-6+7i)=(x(2-3i),x(5+4i),x(-6+7i))$ implies that $12-5i=x(2-3i)$ and $7+22i=x(5+4i)$. Solve both equations for x to see that $(12-5i)/(2-3i)=x=(7+22i)/(5+4i)$. The multiplicative inverses of $2-3i$ and $5+4i$ are $\frac{2}{13}+\frac{3}{13}i$ and $\frac{5}{41}-\frac{4}{41}i$. Therefore, $(12-5i)(\frac{2}{13}+\frac{3}{13}i)=x=(7+22i)(\frac{5}{41}-\frac{4}{41}i)$. Observe $3+2i=\frac{39}{13}+\frac{26}{13}i=(\frac{24}{13}+\frac{15}{13})+(\frac{36}{13}-\frac{10}{13})i=(12-5i)(\frac{2}{13}+\frac{3}{13}i)=x=(7+22i)(\frac{5}{41}-\frac{4}{41}i)=(\frac{35}{41}+\frac{88}{41})+(\frac{110}{41}-\frac{28}{41})i=$~~

Assume, for the purpose of contradicting, that such a complex number $x$ exists. Then $(12-5i,7+22i,-32-9i)=x(2-3i,5+4i,-6+7i)=(x(2-3i),x(5+4i),x(-6+7i))$ implies that $12-5i=x(2-3i)$ and $-32-9i=x(-6+7i)$. Solve both equations for $x$ to see that $(12-5i)/(2-3i)=x=(-32-9i)/(-6+7i)$. The multiplicative inverses of $2-3i$ and $-6+7i$ are $\frac{2}{2^2+3^2}+\frac{3}{2^2+3^2}i=\frac{2}{4+9}+\frac{3}{4+9}i=\frac{2}{13}+\frac{3}{13}i$ and $-\frac{6}{6^2+7^2}-\frac{7}{6^2+7^2}i=-\frac{6}{36+49}-\frac{7}{36+49}i=-\frac{6}{85}-\frac{7}{85}i$ respectively.

~~Observe $3+2i=\frac{39}{13}+\frac{26}{13}i=(\frac{24}{13}+\frac{15}{13})+(\frac{36}{13}-\frac{10}{13})i=(12-5i)(\frac{2}{13}+\frac{3}{13}i)=(12-5i)/(2-3i)=x=(-32-9i)/(-6+7i)=(-32-9i)(-\frac{6}{85}-\frac{7}{85}i)=(+\frac{192}{85}+\frac{63}{85})+(+\frac{224}{85}-\frac{54}{85})i=$ (~~

Observe $3+2i=\frac{39}{13}+\frac{26}{13}i=(\frac{24}{13}+\frac{15}{13})+(\frac{36}{13}-\frac{10}{13})i=(12-5i)(\frac{2}{13}+\frac{3}{13}i)=(12-5i)/(2-3i)=x=(-32-9i)/(-6+7i)=(-32-9i)(-\frac{6}{85}-\frac{7}{85}i)=(\frac{192}{85}-\frac{63}{85})+(\frac{224}{85}+\frac{54}{85})i=\frac{129}{85}+\frac{278}{85}i$. But since $3+2i\neq\frac{129}{85}+\frac{278}{85}i$, we have our desired contradiction. □

Note that if $-32-9i$ had instead been $-32+9i$, $x=3+2i$ would have been a valid solution.

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**12. Show that $(x+y)+z = x+(y+z)$ for all $x,y,z \in \mathbf{F}^n$.**

$(x+y)+z = ((x_1,\dots,x_n)+(y_1,\dots,y_n)) + z = (x_1+y_1,\dots,x_n+y_n)+(z_1,\dots,z_n) =$
$(x_1+y_1+z_1,\dots,x_n+y_n+z_n) = (x_1,\dots,x_n)+(y_1+z_1,\dots,y_n+z_n) =$
$x+((y_1,\dots,y_n)+(z_1,\dots,z_n)) = x+(y+z)$

**13. Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and $a,b \in \mathbf{F}$.**

$(ab)x = (ab)(x_1,\dots,x_n) = (abx_1,\dots,abx_n) = a(bx_1,\dots,bx_n) = a(b(x_1,\dots,x_n)) =$
$a(bx)$

**14. Show that $1x = x$ for all $x \in \mathbf{F}^n$.**

$1x = 1(x_1,\dots,x_n) = (1x_1,\dots,1x_n) = (x_1,\dots,x_n) = x$

**15. Show that $a(x+y) = ax+ay$ for all $a \in \mathbf{F}$ and all $x,y \in \mathbf{F}^n$.**

$a(x+y) = a((x_1,\dots,x_n)+(y_1,\dots,y_n)) = a(x_1+y_1,\dots,x_n+y_n) =$
$(a(x_1+y_1),\dots,a(x_n+y_n)) = (ax_1+ay_1,\dots,ax_n+ay_n) = (ax_1,\dots,ax_n)+(ay_1,\dots,ay_n) =$
$a(x_1,\dots,x_n)+a(y_1,\dots,y_n) = ax+ay$

**16. Show that $(a+b)x = ax+bx$ for all $a,b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.**

~~Show~~ $(a+b)x = (a+b)(x_1,\dots,x_n) = ((a+b)x_1,\dots,(a+b)x_n) = (ax_1+bx_1,\dots,ax_n+bx_n)$
$= (ax_1,\dots,ax_n)+(bx_1,\dots,bx_n) = a(x_1,\dots,x_n)+b(x_1,\dots,x_n) = ax+bx$

# 1.B Definition of Vector Space

## Example 1.24 $F^S$ is a vector space

First, we show that $\mathbf{F}^S$ is commutative. Let $f,g \in \mathbf{F}^S$. Let $x \in S$.
$(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$. Therefore, $f+g = g+f$.

Next, we show that $\mathbf{F}^S$ is associative. Let $f,g,h \in \mathbf{F}^S$ and $a,b \in \mathbf{F}$.
Then ~~$f(x)+g(x)+h(x)$~~ $((f+g)+h)(x) = (f+g)(x)+h(x) = f(x)+g(x)+h(x) =$
$f(x)+(g+h)(x) = (f+(g+h))(x)$ for all $x \in S$. Additionally, $((ab)f)(x) =$
$(ab)f(x) = a(bf(x)) = a(bf)(x) = (a(bf))(x)$. So $(f+g)+h = f+(g+h)$ and $(ab)f = a(bf)$.

# Linear Algebra Done Right 1.B

The additive identiy of the vector space $\mathbf{F}^S$ is $0 \in \mathbf{F}^S$ which is defined by $0(x) = 0$ for all $x \in S$. Let $f \in \mathbf{F}^S$. Then $(f+0)(x) = f(x) + 0(x) = f(x)$ for all $x \in S$. Therefore, $f + 0 = f$.

Now we show that the is an additive inverse for every element of $\mathbf{F}^S$. Let $f \in \mathbf{F}^S$. Choose $-1f$ as the inverse. Then $(f + -1f)(x) = f(x) + (-1f)(x) = f(x) + -1 \cdot f(x) = f(x) - f(x) = 0 = 0(x)$. Therefore, $f + -1f = 0$.

Since for every $f \in \mathbf{F}^S$ and every $x \in S$, $(1f)(x) = 1f(x) = f(x)$, we know that $1f = f$. Therefore, $1$ is a multiplicative identity for the vector space $\mathbf{F}^S$.

Finally, all that remains is to show that $\mathbf{F}^S$ obeys the distributive ~~property~~ properties of a vector space. Let $a, b \in \mathbf{F}$ and $f, g \in \mathbf{F}^S$. ~~Then a(f+g)~~ Let $x \in S$. Then $(a(f+g))(x) = a(f+g)(x) = a(f(x)+g(x)) = af(x) + ag(x) = (af)(x) + (ag)(x) = (af+ag)(x)$. So $a(f+g) = af + ag$. Additionally, $((a+b)f)(x) = (a+b)f(x) = af(x) + bf(x) = (af)(x) + (bf)(x) = (af+bf)(x)$, which implies $(a+b)f = af + bf$.

Based on all the properties proved above, we know that $\mathbf{F}^S$ is a vector space. □

**1. Prove that $-(-v) = v$ for every $v \in V$.**

$-(-v) = -1(-1v) = (-1 \cdot -1)v = 1v = v$

**2. Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.**

If $a = 0$, then we would have our desired. Therefore, we may assume $a \neq 0$. Then $v = 1v = (\frac{a}{a})v = (\frac{1}{a} \cdot a)v = \frac{1}{a}(av) = (\frac{1}{a})0 = 0$.

**3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.**

Let $x, y \in V$ such that $v + 3x = w$ and $v + 3y = w$. We must show that $x = y$. ~~Since~~ Notice that $v + 3x = w = v + 3y$. Add $-v$ to both sides to get $3x = 3y$. Then $x = 1x = (\frac{1}{3} \cdot 3)x = \frac{1}{3}(3x) = \frac{1}{3}(3y) = (\frac{1}{3} \cdot 3)y = 1y = y$.

4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Additive identity. There is no element $0 \in \varnothing$ such that $v+0=v$ for all $v \in \varnothing$.

6. Let $\infty$ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ such that the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t<0 \\ 0 & \text{if } t=0 \\ \infty & \text{if } t>0 \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t<0 \\ 0 & \text{if } t=0 \\ -\infty & \text{if } t>0 \end{cases} \qquad \begin{aligned} t+\infty &= \infty+t = \infty \\ t+(-\infty) &= (-\infty)+t = -\infty \\ \infty+\infty &= \infty \end{aligned}$$

$$(-\infty)+(-\infty) = -\infty \qquad \infty+(-\infty) = 0$$

Is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$?

No. The distributive property does not hold because $(-1+2)\infty = 1\infty = \infty$ but $(-1)\infty + 2\infty = -\infty + \infty = 0$. The associative property does not hold because $(\infty+(-\infty))+1 = 0+1 = 1$ but $\infty + ((-\infty)+1) = \infty + -\infty = 0$.

# 1.C Subspaces

Example 1.35 Verify the following assertions.

a. If $b \in \mathbf{F}$, then $\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$ is a subspace of $\mathbf{F}^4$ if and only if $b=0$.

Let $b \in \mathbf{F}$ and let $X = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$.

First, we show that if $X$ is a subspace of $\mathbf{F}^4$, then $b=0$. Assume that $X$ is a subspace of $\mathbf{F}^4$. Notice that $(0,0,b,0) \in X$ because $b = 5 \cdot 0 + b$. Since $X$ is closed under scalar multiplication, $2(0,0,b,0) = (0,0,2b,0) \in X$. By the definition of $X$, this implies $2b = 5 \cdot 0 + b = b$. Subtract $b$ from both sides to obtain $b=0$.

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Next, we show that if $b=0$, then $X$ is a subspace of $\mathbf{F}^4$. We must show that three conditions are met, assuming $b=0$.

1. ~~0 ∈ X~~ $X$ includes the additive identity, $0$. We know $0=(0,0,0,0)=(0,0,b,0)\in X$ by the definition of $X$ since $0\in\mathbf{F}^4$ and $b=5\cdot 0+b$.

2. $X$ is closed under addition. Assume that $u,w\in X$. Then $u=(u_1,u_2,u_3,u_4)\in X$ implies $u_3=5u_4+b$. Similarly, $w=(w_1,w_2,w_3,w_4)\in X$ tells us that $w_3=5w_4+b$. Since $b=0$, we can simplify those equations to $u_3=5u_4$ and $w_3=5w_4$. Observe that $u+w=(u_1,u_2,u_3,u_4)+(w_1,w_2,w_3,w_4)=(u_1+w_1,u_2+w_2,u_3+w_3,u_4+w_4)=(u_1+w_1,u_2+w_2,5u_4+5w_4,u_4+w_4)=(u_1+w_1,u_2+w_2,5(u_4+w_4)+0,u_4+w_4)=(u_1+w_1,u_2+w_2,5(u_4+w_4)+b,u_4+w_4)\in X$.

3. $X$ is closed under scalar multiplication. Assume that $a\in\mathbf{F}$ and $u\in X$. Then $u=(u_1,u_2,u_3,u_4)$ implies that $u_3=5u_4+b=5u_4+0=5u_4$. So $au=a(u_1,u_2,u_3,u_4)=(au_1,au_2,au_3,au_4)$ tells us that all we need to show that $au\in X$, is to show that $au_3=5(au_4)+b$. Observe $au_3=a(5u_4)=5(au_4)+0=5(au_4)+b$. We have our desired result. $\square$

**b. The set of continuous real-valued functions on the interval $[0,1]$ is a subspace of $\mathbf{R}^{[0,1]}$.**

We must show that three conditions are met to show that the set $S=\{f\in\mathbf{R}^{[0,1]} : f \text{ is continuous}\}$ is a subspace.

First, we know that the additive identity $0\in\mathbf{R}^{[0,1]}$ is in $S$, because $0(x)=0$ for all $x\in\mathbf{R}$ which means $0$ is continuous.

Next, we know that $S$ is closed under addition because we know from calculus that the sum of two continuous functions is continuous.

Finally, we know that $S$ is closed under scalar multiplication because the constant multiple of a continuous function is continuous.

**c. The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.**

Since $0(x)=0$ for all $x\in\mathbf{R}$, $0$ is a constant function. Therefore $0$ is differentiable, which means it's included in the set. Then the set contains an additive identity. Since the set of differentiable real-valued functions is also closed under addition and scalar (constant) multiplication, it's a subspace of $\mathbf{R}^{\mathbf{R}}$.

**d. The set of differentiable real-valued functions $f$ on the interval $(0,3)$ such that $f'(2)=b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b=0$.**

Let $S=\{f\in\mathbf{R}^{(0,3)}: f \text{ is differentiable and } f'(2)=b\}$.

First, we show that if $S$ is a subspace of $\mathbf{R}^{(0,3)}$, then $b=0$. $S$ contains an additive identity $0$ defined by $0(x)=0$ for all $x\in\mathbf{R}^{(0,3)}$. By the constant rule, $b=0'(2)=0$.

Next, we show that if $b=0$, then $S$ is a subspace of $\mathbf{R}^{(0,3)}$. Consider $0\in\mathbf{R}^{(0,3)}$ defined by $0(x)=0$ for all $x\in(0,3)$. Since $0$ is a constant function, $0$ is differentiable and $0'(2)=0=b$. Therefore $0\in S$. Now we show that $S$ is closed under addition. Let $f,g\in S$. Since $f$ and $g$ are differentiable, $f+g$ is differentiable. Also, $(f+g)'(2)=f'(2)+g'(2)=b+b=0+0=0=b$, so $(f+g)\in S$. Finally, we show that $S$ is closed under scalar multiplication. Let $f\in S$ and let $a\in\mathbf{R}$. Since $f$ is differentiable, $af$ is differentiable. Additionally, $(af)'(2)=af'(2)=ab=a\cdot 0=0=b$. Then $af\in S$. Therefore, $S$ is a subspace of $\mathbf{R}^{(0,3)}$.

**e. The set of all sequences of complex numbers with limit $0$ is a subspace of $\mathbf{C}^\infty$.**

Let $S$ be the set of all sequences with limit $0$. We show that $S$ meets the three criteria of a subspace.

First, we show that $S$ includes the additive identity $0$. Since $0=(0,0,\dots)$ is an infinite sequence of the constant $0$, we know the limit of the sequence is that constant, $0$. Therefore, $0\in S$.

Next, we show that $S$ is closed under addition. Let $u,v\in S$. Then $\lim\limits_{n\to\infty}(u_n+v_n)=\lim\limits_{n\to\infty}u_n+\lim\limits_{n\to\infty}v_n=0+0=0$. Therefore, $u+v\in S$.

Finally, we show $S$ is closed under scalar multiplication. Let $u\in S$ and $a\in\mathbf{C}$. Then

$\lim_{n\to\infty} au_n = a \lim_{n\to\infty} u_n = a\cdot 0 = 0$. So by all three of these propeties, we know that $S$ is a subspace of $\mathbf{C}^\infty$.

**Example 1.41** Let $U=\{(x,y,0)\in \mathbf{F}^3 : x,y\in\mathbf{F}\}$ and $\{(0,0,z)\in\mathbf{F}^3 : z\in\mathbf{F}\}$. Show that $\mathbf{F}^3 = U\oplus W$.

Observe $U+W=\{(x,y,0)+(0,0,z)\in \mathbf{F}^3 : x,y,z\in\mathbf{F}\} = \{(x,y,z)\in\mathbf{F}^3 : x,y,z\in\mathbf{F}\}$.

First, we show that $\mathbf{F}^3 = U+W$. From above, we see that $U+W\subseteq\mathbf{F}^3$. We show that $\mathbf{F}^3\subseteq U+W$. Let $x=(x_1,x_2,x_3)\in\mathbf{F}^3$. Since $x_1,x_2,x_3\in\mathbf{F}$, we know that $(x_1,x_2,x_3)=x\in\{(x,y,z)\in\mathbf{F}^3 : x,y,z\in\mathbf{F}\}=U+W$. So $\mathbf{F}^3\subseteq U+W$ which, together with $U+W\subseteq\mathbf{F}^3$, implies $\mathbf{F}^3=U+W$.

~~Next, we show that $U+W$ is a direct sum. Let $u_1,u_2\in U$ and $w_1,w_2\in W$ such that $u_1+w_1=u_2+w_2$. We must show that $u_1=u_2$ and $w_1=w_2$. Observe~~

Next, we show that $U+W$ is a direct sum. Let $a,b\in U$ and $x,y\in W$ such that $a+x=b+y$. We must show that $a=b$ and $x=y$. Observe $a+x=(a_1,a_2,0)+(0,0,x_3)=(a_1,a_2,x_3)$ and $b+y=(b_1,b_2,0)+(0,0,y_3)=(b_1,b_2,y_3)$. Then $(a_1,a_2,x_3)=a+x=b+y=(b_1,b_2,y_3)$ which implies $a_1=b_1$, $a_2=b_2$ and $x_3=y_3$. Therefore, $a=(a_1,a_2,0)=(b_1,b_2,0)=b$ and $x=(0,0,x_3)=(0,0,y_3)=y$.

# Exercises 1.C

**1** **For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.**

**a.** $\{(x_1,x_2,x_3)\in\mathbf{F}^3 : x_1+2x_2+3x_3=0\}$

Let $A=\{(x_1,x_2,x_3)\in\mathbf{F}^3 : x_1+2x_2+3x_3=0\}$. We must show that $A$ meets the three conditions for a subspace.

First, we show that $A$ has the additive identity of $\mathbf{F}^3$, $0=(0,0,0)$. Since $0+2\cdot0+3\cdot0=0$, we know that $(0,0,0)=0\in A$.

Next, we show that $A$ is closed under addition. Let $u,v\in A$.

Then $u_1 + 2u_2 + 3u_3 = 0 = v_1 + 2v_2 + 3v_3$. ~~So 0 = 0+0~~ observe $u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1+v_1, u_2+v_2, u_3+v_3)$. Additionally, $0 = 0 + 0 = u_1 + 2u_2 + 3u_3 + v_1 + 2v_2 + 3v_3 = (u_1+v_1) + (2u_2+2v_2) + (3u_3+3v_3) = (u_1+v_1) + 2(u_2+v_2) + 3(u_3+v_3)$. Therefore, $u + v \in A$.

Finally, we show that $A$ is closed under scalar multiplication. Let $a \in \mathbf{F}$ and $u \in A$. Then $u_1 + 2u_2 + 3u_3 = 0$. Multiply both side by $a$ to see that $0 = a(u_1 + 2u_2 + 3u_3) = au_1 + 2au_2 + 3au_3$. Since $au = a(u_1, u_2, u_3) = (au_1, au_2, au_3)$, we know that $au \in A$.

b. Let $B = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

$B$ is not a subspace of $\mathbf{F}^3$. Since $0 + 2 \cdot 0 + 3 \cdot 0 = 0 \neq 4$, we know that $(0,0,0) = 0 \notin B$. Therefore, $B$ does not meet the additive identity condition of a subspace.

c. Let $C = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$

$C$ is not a subspace because it does not meet the closed under addition condition of a subspace. Consider $(1,0,0) + (0,1,1) = (1,1,1)$. Since $1 \cdot 0 \cdot 0 = 0 = 0 \cdot 1 \cdot 1$, we know $(1,0,0), (0,1,1) \in C$. But since $1 \cdot 1 \cdot 1 = 1 \neq 0$, we know $(1,1,1) \notin C$.

d. Let $D = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

$D$ is a subspace of $\mathbf{F}^3$. We show that $D$ meets the three conditions of a subspace.

First, we show that $D$ has the additive identity of $\mathbf{F}^3$, $0$. Since $0 = 5 \cdot 0$, we know $(0,0,0) = 0 \in D$.

Second, we show that $D$ is closed under addition. Let $u, v \in D$. Then $u_1 = 5u_3$ and $v_1 = 5v_3$. Note that $u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1+v_1, u_2+v_2, u_3+v_3)$. Since $u_1 + v_1 = 5u_3 + 5v_3 = 5(u_3+v_3)$, we know that $u + v \in D$.

Third, we show $D$ is closed under scalar multiplication. Let $u \in D$. Then $u_1 = 5u_3$. Since $au = (au_1, au_2, au_3)$ and $au_1 = a(5u_3) = 5au_3$, we know $au \in D$.

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3. Show that the set of differentiable real valued functions $f$ on the interval $(-4,4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

Let $S$ be the set of such functions. We show that $S$ meets the three conditions of a subspace. Let $x$ be an arbitrary element of $(-4,4)$.

First, we show that $S$ has an additive identity. Let $0 : (-4,4) \to \mathbf{R}$ be defined by $0(x) = 0$. Let $f \in S$. Then $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$. Therefore, $f + 0 = f$. Since $0$ is a constant function, $0'(-1) = 0 = 3 \cdot 0 = 3f(2)$, which implies $0 \in S$.

Next, we show that $S$ is closed under addition. Let $f, g \in S$. Then $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$. Then $(f+g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f(2) + g(2)) = 3(f+g)(2)$. Therefore, $f + g \in S$.

Finally, we show that $S$ is closed under scalar multiplication. Let $a \in \mathbf{R}$ and let $f \in S$. Then $f'(-1) = 3f(2)$. So $(af)'(-1) = af'(-1) = a(3f(2)) = 3af(2) = 3(af)(2)$. Therefore, $af \in S$.

5. Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

~~No because $\mathbf{R}^2$ is not cl Yes, we show that $\mathbf{R}^2$ meets the three conditions of a subspace~~

No, because $\mathbf{R}^2$ is not closed under complex scalar multiplication. Consider $i(1,1) = (i,i)$. Then $i \in \mathbf{C}$ and $(1,1) \in \mathbf{R}^2$ but $(i,i) \notin \mathbf{R}^2$.

6a. Is $\{(a,b,c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?

Call the above set $S$. $S$ is a subspace of $\mathbf{R}^3$. Since $0^3 = 0^3$, we know that $(0,0,0) = 0 \in S$. Let $a \in \mathbf{R}$ and let $u, v \in S$. Then $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ for some $u_1, v_1, \dots, u_3, v_3 \in \mathbf{R}$. Since $u, v \in S$, we know that $u_1^3 = u_2^3$ and $v_1^3 = v_2^3$. Therefore, $u_1 = u_2$ and $v_1 = v_2$. So $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$. Since $(u_1 + v_1)^3 = (u_2 + v_2)^3$, we know that $u + v \in S$. Also $au = a(u_1, u_2, u_3) = (au_1, au_2, au_3)$. Since $(au_1)^3 = (au_2)^3$, we know that $au \in S$.

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**6b. Let $S=\{(a,b,c)\in\mathbf{C}^3 : a^3=b^3\}$. Is $S$ a subspace of $\mathbf{C}^3$?**

No, because $S$ is not closed under addition. For example, consider $\left(1,\frac{-1+\sqrt{3}i}{2},0\right)+\left(1,\frac{-1+\sqrt{3}i}{2},0\right)=(2,-1,0)$. The first term is in $S$ because $1^3=1=\left(\frac{-1+\sqrt{3}i}{2}\right)^3$. The second term is in $S$ because $1^3=1=\left(\frac{-1-\sqrt{3}i}{2}\right)^3$. However, the sum, $(2,-1,0)$, is not in $S$ because $2^3=8\neq-1=(-1)^3$.

**7. Give an example of a nonempty set $U\subseteq\mathbf{R}^2$ such that $U$ is closed under addition and under taking additive inverses (meaning $-u\in U$ whenever $u\in U$), but $U$ is not a subspace of $\mathbf{R}^2$.**

~~Let $U=\{(x,y)\in\mathbf{R}^2$ : x and y are to~~

Let $U=\{(2x,2y)\in\mathbf{R}^2 : x \text{ and } y \text{ are integers}\}$. We show that $U$ is closed under addition. Let $(2a,2b),(2c,2d)\in U$ where $a,b,c,d\in\mathbf{Z}$. Then $(2a,2b)+(2c,2d)=(2a+2c,2b+2d)=(2(a+c),2(b+d))$. Since $a+c$ and $b+d$ are both integers, $(2a,2b)+(2c+2d)\in U$. Additionally, since $-(2a,2b)=(-2a,-2b)=(2(-a),2(-b))$ and $-a$ and $-b$ are integers, $-(2a,2b)\in U$. Therefore, $U$ is closed under taking additive inverses. However, $U$ is not a subspace of $\mathbf{R}^2$ because it is not closed under scalar multiplication. Consider $\frac{1}{2}(2,2)=(1,1)$. We know $(2,2)=(2\cdot1,2\cdot1)\in U$, but $(1,1)\notin U$. Note, in hindsight $\{(x,y)\in\mathbf{R}^2 : x,y\in\mathbf{Z}\}$ would've been a simpler example

**8. Give an example of a nonempty subset $U$ of $\mathbf{R}^2$ such that $U$ is closed under scalar multiplication, but $U$ is not a subspace of $\mathbf{R}^2$.**

~~Let $U=\{(x,y)\in\mathbf{R}^2 : \forall\in\mathbf{R}\}$~~ Let $U=\{(x,y)\in\mathbf{R}^2 : x^2=y^2\}$. We show that $U$ is closed under scalar multiplication. Let $(x,y)\in U$. Then $x^2=y^2$. Let $a\in\mathbf{R}$. Then $a(x,y)=(ax,ay)$. Since $(ax)^2=a^2x^2=a^2y^2=(ay)^2$, we know that $a(x,y)\in U$. However, $U$ is not a subspace of $\mathbf{R}^2$ because $U$ is not closed under addition. Consider $(1,1)+(-1,1)=(0,2)$. We know $(1,1),(-1,1)\in U$ because $1^2=1^2=(-1)^2=1$. But $(0,2)\notin U$ because $0^2=0\neq4=2^2$.

**9. A function $f: \mathbf{R} \to \mathbf{R}$ is called periodic if there exists a positive number $p$ such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$?**

Fix $x$ as an arbitrary real number for this entire proof. The set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ is not a subspace because it's not closed under addition.

Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \cos 2\pi x$. Recall that cos is periodic with a period of $2\pi$. We show that $f$ is periodic. Observe $f(x+1) = \cos 2\pi(x+1) = \cos(2\pi x + 2\pi) = \cos 2\pi x = f(x)$.

Since cos and $f$ are both periodic functions, $\cos + f$ must be periodic if the set of periodic functions were closed under addition. However, we show that $\cos + f$ is not periodic. That is, there's no positive number $p$ such that $(\cos + f)(x) = (\cos + f)(x+p)$. Assume, for the purpose of contradicting, that such a number $p$ exists. Then $(\cos + f)(0) = (\cos + f)(p)$ which implies $\cos 0 + \cos 2\pi 0 = \cos p + \cos 2\pi p$. Therefore, $\cos p + \cos 2\pi p = 1 + 1 = 2$. Since 1 is the maximum value in the range of cos, we know that $\cos p + \cos 2\pi p = 2$ implies $\cos p = 1 = \cos 2\pi p$. Then we can infer the following: $p = 2x\pi$ and $2\pi p = 2y\pi$ for some pair of integers $x$ and $y$. Solve that second equation for $p$ to get $y = p = 2x\pi$. ~~However, $y \neq 2x\pi$ is a contradiction because $y$ is rational and $2x\pi$ is irrational~~. Divide both sides by $2x$ (note that $2x \neq 0$ because $p$ is positive) to obtain $\pi = \frac{y}{2x}$, which is a contradiction because $\pi$ is irrational and $\frac{y}{2x}$ is rational.

**10. Suppose $U_1$ and $U_2$ are subspaces of $V$. Prove that the intersection $U_1 \cap U_2$ is a subspace of $V$.**

Since $U_1$ and $U_2$ are ~~vector~~ subspaces of $V$, $0 \in U_1$ and $0 \in U_2$. Therefore, $0 \in U_1 \cap U_2$. We show that $U_1 \cap U_2$ is closed under addition. Let $u, v \in U_1 \cap U_2$. Then $u, v \in U_1$ and $u, v \in U_2$. So $u + v \in U_1$ and $u + v \in U_2$. Therefore, $u + v \in U_1 \cap U_2$. We show that $U_1 \cap U_2$ is closed under scalar multiplication. Let $a \in \mathbf{F}$. Then $au \in U_1$ and $au \in U_2$. Therefore, $au \in U_1 \cap U_2$.

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**11. Prove that the intersection of every collection of subspaces of V is a subspace of V.**

~~Let $U_1, \ldots, U_n$~~ Let $n$ be a positive integer. For every integer $1 \le j \le n$, let $U_j$ be a subspace of $V$. We must show that ~~$U_1 \cap \cdots \cap U_n$~~ $U_1 \cap \cdots \cap U_n$ is a subspace of $V$. ~~By exercise 10, we know $U_1 \cap U_2$ is a subspace of $V$. Therefore, the statement is true for $n$~~ Since $U_1$ is a subspace of $V$, we know the statement is true for $n=1$. We proceed via induction. Assume that $U_1 \cap \cdots \cap U_k$ is a subspace of $V$ for some positive integer ~~$k$~~ $k<n$. We must show that $U_1 \cap \cdots \cap U_{k+1}$ is a subspace of $V$. By exercise 10, we know that $(U_1 \cap \cdots \cap U_k) \cap U_{k+1} =$ ~~$U_1 \cap \cdots \cap U_{k+1}$~~ $U_1 \cap \cdots \cap U_{k+1}$ is a subspace of $V$. $\square$

**12. Prove that the union of two subspaces of V is a subspace if and only if one of the subspaces is contained in the other.**

Let $U$ and $W$ be subspaces of $V$. We must show that $U \cup W$ is a subspace of $V$ if and only if $U \subseteq W$ or $W \subseteq U$.

First, we show that if $U \subseteq W$ or $W \subseteq U$, then $U \cup W$ is a subspace of $V$. If $U \subseteq W$, then $U \cup W = W$ which is a subspace of $V$. If $W \subseteq U$, then $U \cup W = U$ which is a subspace of $V$.

Next, we show the converse: if $U \cup W$ is a subspace of $V$, then $U \subseteq W$ or $W \subseteq U$. Assume that $U \cup W$ is a subspace. If $U \subseteq W$, then we have our result. Therefore, we may assume $U \not\subseteq W$. Then there must be some $u \in U$ such that $u \notin W$. Let $w \in W$. Then $u + w \in U \cup W$. So $u + w \in U$ or $u + w \in W$. But if $u + w \in W$, then $u = (u+w) - w \in W$ which contradicts $u \notin W$. Therefore, $u + w \in U$. Then $w = (u+w) - u \in U$. Since $w \in U$ for every $w \in W$, we know that $W \subseteq U$.

**13. Prove that the union of three subspaces of V is a subspace if and only if one of the subspaces contains the other two.**

First, we show that if one subspace of $V$ contains two subspaces of $V$, then the union of all three subspaces is a subspace of $V$. Let $R, S, T$ be three subspaces of $V$ such that $S \subseteq R$ and $T \subseteq R$. Then $R \cup S \cup T = R$ which is a subspace of $V$.

Now we show the converse: if the union of three subspaces of $V$ is a subspace of $V$, then one of the subspaces contains the other two. Let $S_1$, $S_2$, and $S_3$ be subspaces of $V$. Assume that $\bigcup_{i=1}^{3} S_i$ is a subspace. We must show that for some ~~every~~ integer $1 \le j \le 3$, $\bigcup_{i=1}^{3} S_i \subseteq S_j$. We'll assume the converse and produce a contradiction, so assume that for every integer $1 \le j \le 3$, $\bigcup_{i=1}^{3} S_i \not\subseteq S_j$. Since $\bigcup_{i=1}^{3} S_i \not\subseteq S_1$, we know that there's some $x \in \bigcup_{i=1}^{3} S_i$ such that $x \notin S_1$. Let $y \in S_1$. Then there are infinite vectors of the form $y + kx$, where $k \in \mathbf{F}$ and $k \neq 0$ which are elements of $\bigcup_{i=1}^{3} S_i$.

If $y + kx \in S_1$, then $x = \frac{k}{k}x = \frac{1}{k}(kx) = \frac{1}{k}((y+kx) - y) \in S_1$ which contradicts $x \notin S_1$. Therefore, $y + kx \notin S_1$. So $y + kx \in S_2 \cup S_3$. Since there are infinite such vectors, then there are certainly more ~~that~~ than two. Therefore by the pigeonhole principle, there must be some $n \in \{2, 3\}$ such that $S_n$ contains more than one such vectors. So $y + k_1 x, y + k_2 x \in S_n$ for some distinct $k_1, k_2 \in \mathbf{F} \setminus \{0\}$. Observe $(k_1 - k_2)x = k_1 x - k_2 x = k_1 x - k_2 x + y - y = (y + k_1 x) - (y + k_2 x) \in S_n$. Therefore, $x \in S_n$. Then $y = (y + k_1 x) - k_1 x \in S_n$. So since $y \in S_n$ for every $y \in S_1$, we know that $S_1 \subseteq S_n$.

With that in mind, notice how $\bigcup_{i=1}^{3} S_i = (S_1 \cup S_n) \cup S_m$ for some integer $1 \le m \le 3$ such that $m \notin \{1, n\}$. Therefore, $(S_1 \cup S_n) \cup S_m$ being a subspace implies that $S_n \cup S_m$ is a subspace. By problem 12 we know that one of those subspaces contains the other let $r$ be the integer such that $S_r$ is the containing subspace. Then $\bigcup_{i=1}^{3} S_i \subseteq S_r$, which is a contradiction.

**14. Suppose $U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$ and $W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}$. Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.**

$U + W = \{(x+y, -x+y, 2(x+y)) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}$. Furthermore, we'll show that is equivalent to $\{(a, b, 2a) \in \mathbf{F}^3 : a, b \in \mathbf{F}\}$. First, we show that if $x, y \in \mathbf{F}$, then $(x+y, -x+y, 2(x+y)) = (a, b, 2a)$ for some $a, b \in \mathbf{F}$. Choose $a = x + y$ and $b = -x + y$ and we have the desired result.

Next, we show the converse: if $a, b \in \mathbf{F}$, then $(a, b, 2a) = (x+y, -x+y, 2(x+y))$ for some $x, y \in \mathbf{F}$. Choose $x = \frac{a-b}{2}$ and $y = \frac{a+b}{2}$. Then observe the following:

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$(x+y, -x+y, 2(x+y)) = \left(\frac{a-b}{2} + \frac{a+b}{2}, -\frac{a-b}{2} + \frac{a+b}{2}, \left(\frac{a-b}{2} + \frac{a+b}{2}\right) \cdot 2\right) = \left(\frac{2a}{2}, \frac{-a+b}{2} + \frac{a+b}{2}, 2\left(\frac{2a}{2}\right)\right) = \left(a, \frac{2b}{2}, 2a\right) = (a, b, 2a).$

Therefore, $\{(x+y, -x+y, 2(x+y)) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} = \{(a, b, 2a) \in \mathbf{F}^3 : a, b \in \mathbf{F}\}$.

From this we see that $U+W$ is the plane containing the y-axis and the line $z=2x$. Or another way, $U+W$ is the plane with normal $(2,0,-1)$.

**15. Suppose that $U$ is a subspace of $V$. What is $U+U$?**

By theorem 1.40, $U+U$ is the smallest subspace containing both $U$ and, well... $U$. Therefore, $U+U=U$.

**16. Is the operation of addition on the subspaces of $V$ commutative?**

Again, this is clear to see in light of theorem 1.40. Let $U$ and $W$ be subspaces of $V$. Then $U+W$ is the smallest subspace that contains $U$ and $W$. Furthermore, $W+U$ is the smallest subspace that contains both $W$ and $U$. Since logical "and" is commutative, it follows that those are equivalent statements so $U+W=W+U$.

**17. Is the operation of addition on the subspaces of $V$ associative?**

Yes, another fact which logically follows from theorem 1.40.

**18. Does the operation of addition on the subspaces of $V$ have an additive identity? Which subspaces have additive inverses?**

Yes, $\{0\}$ is the additive identity for addition over the subspaces of $V$. Let $U$ be a subspace of $V$. Then $0 \in U$. Therefore, $U$ is the smallest subspace that contains both $U$ and $\{0\}$, so $U+\{0\}=U$.

$\{0\}$ is the only subspace with an additive inverse (itself) because $\{0\}+\{0\}=\{0\}$. No other subspace has an additive inverse because theorem 1.40 tells us that subspace addition will result in a larger or same size subspace.

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19. Prove or give a counterexample: If $V_1, V_2, U$ are subspaces of $V$ such that $V_1 + U = V_2 + U$, then $V_1 = V_2$.

False, we show a counterexample. Let $U = \{(x,0) \in \mathbf{F}^2 : x \in \mathbf{F}\}$, $V_1 = \{(0,y) \in \mathbf{F}^2 : y \in \mathbf{F}\}$, and $V_2 = \{(z,z) \in \mathbf{F}^2 : z \in \mathbf{F}\}$. Then $V_1 + U = \{(a,b) \in \mathbf{F}^2 : a,b \in \mathbf{F}\} = \mathbf{F}^2$. Also $V_2 + U = \{v + u : v \in V_2 \text{ and } u \in U\} = \{(z,z) + (x,0) : z,x \in \mathbf{F}\} = \{(x+z,z) : z,x \in \mathbf{F}\}$.

We show that $V_2 + U = \{(x+z,z) : x,z \in \mathbf{F}\} = \{(a,b) \in \mathbf{F}^2 : a,b \in \mathbf{F}\} = \mathbf{F}^2 = U + V_1$. First, we show that if $x,z \in \mathbf{F}$, then $(x+z,z) = (a,b)$ for some $a,b \in \mathbf{F}$. Choose $a = x+z$ and $b = z$. Then $(a,b) = (x+z,z)$.

Next, we show the converse: if $a,b \in \mathbf{F}$, then $(a,b) = (x+z,z)$ for some $x,z \in \mathbf{F}$. Choose $x = a-b$ and $z = b$. Then $(x+z,z) = (a-b+b,b) = (a,b)$. Therefore, $V_2 + U = \{(x+z,z) : x,z \in \mathbf{F}\} = \{(a,b) : a,b \in \mathbf{F}\} = \mathbf{F}^2 = V_1 + U$.

However, $(0,1) \in V_1$ and $(0,1) \in V_2$. Therefore, $V_1 \neq V_2$.

20. Suppose $U = \{(x,x,y,y) \in \mathbf{F}^4 : x,y \in \mathbf{F}\}$. Find a subspace $W$ of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

Let $W = \{(z,-z,w,-w) \in \mathbf{F}^4 : z,w \in \mathbf{F}\}$. Then $U + W = \{u + w : u \in U \text{ and } w \in W\} = \{(x,x,y,y) + (z,-z,w,-w) : x,y,z,w \in \mathbf{F}\} = \{(x+z, x-z, y+w, y-w) : x,y,z,w \in \mathbf{F}\}$. We show that set is equivalent to $\mathbf{F}^4 = \{(a_1,a_2,a_3,a_4) : a_1,\dots,a_4 \in \mathbf{F}\}$.

First, we show that if $x,y,z,w \in \mathbf{F}$, then $(x+z,x-z,y+w,y-w) = (a_1,a_2,a_3,a_4)$ for some $a_1,\dots,a_4 \in \mathbf{F}$. Naturally, we can choose $a_1 = x+z$, $a_2 = x-z$, $a_3 = y+w$, and $a_4 = y-w$ to make the equation hold.

Next, we show that if $a_1,\dots,a_4 \in \mathbf{F}$, then $(a_1,a_2,a_3,a_4) = (x+z,x-z,y+w,y-w)$ for some $x,y,z,w \in \mathbf{F}$. Choose $x = \frac{a_1+a_2}{2}$, $y = \frac{a_3+a_4}{2}$, $z = \frac{a_1-a_2}{2}$, and $w = \frac{a_3-a_4}{2}$. Then observe

$$(x+z,x-z,y+w,y-w) = \left(\frac{a_1+a_2}{2} + \frac{a_1-a_2}{2}, \frac{a_1+a_2}{2} - \frac{a_1-a_2}{2}, \frac{a_3+a_4}{2} + \frac{a_3-a_4}{2}, \frac{a_3+a_4}{2} - \frac{a_3-a_4}{2}\right) = \left(\frac{2a_1}{2}, \frac{2a_2}{2}, \frac{2a_3}{2}, \frac{2a_4}{2}\right) = (a_1,a_2,a_3,a_4).$$

Therefore, $U + W = \{(x+z,x-z,y+w,y-w) : x,y,z,w \in \mathbf{F}\} = \{(a_1,a_2,a_3,a_4) : a_1,\dots,a_4 \in \mathbf{F}\} = \mathbf{F}^4$.

Now we just need to show that $U+W$ is a direct sum, which is the case if and only if $U \cap W = \{0\}$. Let $u \in U$ and ~~$w \in W$~~ $u \in W$. Then $u = (x,x,y,y) = (z,-z,w,-w)$ for some $x,y,z,w \in \mathbf{F}$. Therefore, we know that $x = z$ and $x = -z$, so $z = x = -z$ which implies $z = 0$. Similarly, we see that $w = y = -w$ so $w = 0$. Therefore, $u = (z,-z,w,-w) = (0,0,0,0) = 0$. It follows that $U \cap W = \{0\}$ which implies $U+W$ is a direct sum.

21. Suppose $U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5 : x,y \in \mathbf{F}\}$. Find a subspace $W$ of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Let $W = \{(a,b,0,0,c) : a,b,c \in \mathbf{F}$. Then $U+W = \{u+w : u \in U \text{ and } w \in W\} =$ $\{(x,y,x+y,x-y,2x) + (a,b,0,0,c) : x,y,a,b,c \in \mathbf{F}\} =$ $\{(x+a, y+b, x+y, x-y, 2x+c) : x,y,a,b,c \in \mathbf{F}\}$. We show $U+W = \mathbf{F}^5$ by mutual inclusion. It follows directy that if $v \in U+W$, then $v \in \mathbf{F}^5$. Let ~~me~~ $s \in \mathbf{F}^5$. Then $s = (s_1, s_2, s_3, s_4, s_5)$ for some $s_1, \ldots, s_5 \in \mathbf{F}$. We must show that there exist some $x,y,a,b,c \in \mathbf{F}$ such that $(s_1,s_2,s_3,s_4,s_5) =$ $(x+a, y+b, x+y, x-y, 2x+c)$. Choose $x = \frac{s_3+s_4}{2}$, $y = \frac{s_3-s_4}{2}$, $a = \frac{2s_1-s_3-s_4}{2}$, $b = \frac{2s_2-s_3+s_4}{2}$, and $c = s_5 - s_3 - s_4$. Observe

$$(x+a,\ y+b,\ x+y,\ x-y,\ 2x+c)$$

$$= \left(\frac{s_3+s_4}{2} + \frac{2s_1-s_3-s_4}{2},\ \frac{s_3-s_4}{2} + \frac{2s_2-s_3+s_4}{2},\ \frac{s_3+s_4}{2} + \frac{s_3-s_4}{2},\ x-y,\ 2x+c\right)$$

$$= \left(\frac{2s_1}{2}, \frac{2s_2}{2}, \frac{2s_3}{2}, \frac{s_3+s_4}{2} - \frac{s_3-s_4}{2}, 2\left(\frac{s_3+s_4}{2}\right) + s_5 - s_3 - s_4\right)$$

$$= \left(s_1, s_2, s_3, \frac{s_3+s_4}{2} + \frac{-s_3+s_4}{2}, s_3+s_4+s_5-s_3-s_4\right)$$

$$= \left(s_1, s_2, s_3, \frac{2s_4}{2}, s_5\right) = (s_1, s_2, s_3, s_4, s_5).$$

Therefore $s \in U+W$.

Now to show that $U+W$ is a direct sum, we show that $U \cap W = \{0\}$. Let $v \in U \cap W$. Then $v \in U$ and $v \in W$. So $(x,y,x+y,x-y,2x) = v = (a,b,0,0,c)$ for some $x,y,a,b,c \in \mathbf{F}$. Therefore, $x+y = 0 = x-y$. Subtract $x$ from both sides to see that $y = -y$ which implies $y = 0$. Then $0 = x+y = x+0 = x$. Therefore, $v = (x,y,x+y,x-y,2x) = (0,0,0,0,0) = 0$.

# Linear Algebra Done Right 1.C

**22.** Suppose $U = \{(x, y, x+y, x-y, 2x) : x, y \in \mathbf{F}\}$. Find three subspaces $W_1, W_2, W_3$ of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Let $W_1 = \{(a,0,0,0,0) : a \in \mathbf{F}\}$, $W_2 = \{(0,b,0,0,0) : b \in \mathbf{F}\}$, and $W_3 = \{(0,0,0,0,c) : c \in \mathbf{F}\}$. Then $U + W_1 + W_2 + W_3 = \{(x+a, y+b, x+y, x-y, 2x+c) : a,b,c,x,y \in \mathbf{F}\}$. ~~We show that $U+W_1+W_2+W_3 = \mathbf{F}^5$ by mutual inclusion. It follows directly that every element of $U+W_1+W_2+W_3$ is in $\mathbf{F}^5$. Let $v \in \mathbf{F}^5$. Then $v = (v_1, v_2, v_3, v_4, v_5)$ for some~~ Then by exercise 21, $U + W_1 + W_2 + W_3 = \mathbf{F}^5$.

We show that $U + W_1 + W_2 + W_3$ is a direct sum. Let $u \in U$, $w_1 \in W_1$, $w_2 \in W_2$, and $w_3 \in W_3$. Then $u = (x, y, x+y, x-y, 2x)$, $w_1 = (a,0,0,0,0)$, $w_2 = (0,b,0,0,0)$, and $w_3 = (0,0,0,0,c)$ for some $a,b,c,x,y \in \mathbf{F}$. Assume that $u + w_1 + w_2 + w_3 = 0$. Then by theorem 1.45, if we can show that $0 = u = w_1 = w_2 = w_3$, we'll have the desired result. Observe $0 = u + w_1 + w_2 + w_3 = (x, y, x+y, x-y, 2x) + (a,0,0,0,0) + (0,b,0,0,0) + (0,0,0,0,c) = (x+a, y+b, x+y, x-y, 2x+c)$. Then $x+y = 0 = x-y$. Subtract $x$ from both sides to get $y = -y$ which implies $y = 0$. Then $0 = x+y = x+0 = x$. Additionally, we have $0 = x+a = 0+a = a$, $0 = y+b = 0+b = b$, and $0 = 2x+c = 2\cdot 0 + c = c$. Therefore, $u = (x,y,x+y,x-y,2x) = 0$, $w_1 = (a,0,0,0,0) = 0$, $w_2 = (0,b,0,0,0) = 0$, and $w_3 = (0,0,0,0,c) = 0$.

**23.** Prove or give a counterexample: If $V_1, V_2, U$ are subspaces of $V$ such that $V = V_1 \oplus U$ and $V = V_2 \oplus U$, then $V_1 = V_2$.

~~Let~~ False, we show a counterexample. Let $V_1 = \{(x,0) : x \in \mathbf{F}\}$, $V_2 = \{(y,y) : y \in \mathbf{F}\}$, and $U = \{(0,z) : z \in \mathbf{F}\}$. ~~Let $v \in \mathbf{F}^2$. We show that $\mathbf{F}^2 \subseteq$~~ Observe that $V_1 + U = \{(x,z) : x, z \in \mathbf{F}\} = \mathbf{F}^2$. We also show that $V_2 + U = \mathbf{F}^2$ by mutual inclusion. Since $V_2 + U = \{(y, y+z) : y, z \in \mathbf{F}\}$, we know that every element of $V_2 + U$ is in $\mathbf{F}^2$. Let $v \in \mathbf{F}^2$. Then $v = (v_1, v_2)$ for some $v_1, v_2 \in \mathbf{F}$. We must show that there's some $y, z \in \mathbf{F}$ such that $(v_1, v_2) = (y, y+z)$. Choose $y = v_1$ and $z = -v_1 + v_2$. Then $(y, y+z) = (v_1, v_1 - v_1 + v_2) = (v_1, v_2)$. Therefore, $v \in V_2 + U$.

However, since $(1,0) \in V_1$ but $(1,0) \notin V_2$ we know that $V_1 \neq V_2$.

Linear Algebra Done Right 1.C

24. A function $f: \mathbf{R} \to \mathbf{R}$ is called even if $f(-x) = f(x)$ for all $x \in \mathbf{R}$. A function $f: \mathbf{R} \to \mathbf{R}$ is called odd if $f(-x) = -f(x)$ for all $x \in \mathbf{R}$. Let $V_e$ denote the set of real-valued even functions on $\mathbf{R}$ and let $V_o$ denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^\mathbf{R} = V_e \oplus V_o$.

Fix $x$ as an arbitrary real number for this entire proof.

First, we show that $V_e$ is a subspace of $\mathbf{R}^\mathbf{R}$. Since $0(x) = 0 = 0(-x)$, we know that $0 \in V_e$. Let $f \in V_e$ and $a \in \mathbf{R}$. Since $(af)(-x) = af(-x) = af(x) = (af)(x)$, we know that $af \in V_e$. Now let $g \in V_e$. Since $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, we know $f+g \in V_e$. Therefore, $V_e$ is a subspace of $\mathbf{R}^\mathbf{R}$.

Next we show that $V_o$ is a subspace of $\mathbf{R}$. Observe $0(-x) = 0 = -0 = -0(x)$. Therefore, $0 \in V_o$. Let $f, g \in V_o$. Then $(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f(x) + g(x)) = -(f+g)(x)$. So $f + g \in V_o$. Let $a \in \mathbf{R}$. Since $(af)(-x) = af(-x) = a \cdot -f(x) = -af(x) = -(af)(x)$, we know that $af \in V_o$. Therefore, $V_o$ is a subspace of $\mathbf{R}^\mathbf{R}$.

~~Now we show that $V_e + V_o = \mathbf{R}^\mathbf{R}$ by mutual inclusion. Let $v \in V_e + V_o$. Since $V_e$ and $V_o$ are subspaces of $\mathbf{R}^\mathbf{R}$, we know $V_e + V_o$ is a subspace of $\mathbf{R}^\mathbf{R}$. Therefore, $v \in \mathbf{R}^\mathbf{R}$. Let $h \in \mathbf{R}^\mathbf{R}$. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = h(-x) + h(x)$. Observe $f(-x) = h(-(-x)) + h(-x) = h(x) + h(-x) = h(-x) + h(x) = f(x)$. Therefore, $f$ is even. Let $g$ be defined by $g(x) = h(-x) - h(x)$. Since $g(-x) = h(-(-x)) - h(-x) = -(-(h(x) - h(-x))) = -(-h(x) + h(-x)) = -(h(-x) - h(x)) = -g(x)$, we know that $g$ is odd. ~~Observe $(f+g)(x) = f(x) + g(x)$~~ Observe ~~$h(-x) + h(x)$~~ $(\frac{1}{2}f + \frac{1}{2}g)(x) = (\frac{1}{2}f)(x) + (\frac{1}{2}g)(x) = \frac{1}{2}f(x) + \frac{1}{2}g(x) = \frac{1}{2}(h(-x) + h(x)) + \frac{1}{2}(h(-x) - h(x)) = \frac{1}{2}h(-x) + \frac{1}{2}h(x)$~~

Now we show that $V_e + V_o = \mathbf{R}^\mathbf{R}$ by mutual inclusion. Since $V_e + V_o$ is a subspace of $\mathbf{R}^\mathbf{R}$, we know that $v \in \mathbf{R}^\mathbf{R}$ for all $v \in V_e + V_o$. Let $h \in \mathbf{R}^\mathbf{R}$. Then we know there's some $f, g \in \mathbf{R}^\mathbf{R}$ such that $f$ is defined by $f(x) = \frac{1}{2}h(x) + \frac{1}{2}h(-x)$ and $g$ is defined by $g(x) = \frac{1}{2}h(x) - \frac{1}{2}h(-x)$. Observe $f(-x) = \frac{1}{2}h(-x) + \frac{1}{2}h(-(-x)) = \frac{1}{2}h(-x) + \frac{1}{2}h(x) = \frac{1}{2}h(x) + \frac{1}{2}h(-x) = f(x)$. Therefore, $f$ is even. Since $g(-x) = \frac{1}{2}h(-x) - \frac{1}{2}h(-(-x)) = -(-(\frac{1}{2}h(-x) - \frac{1}{2}h(x))) = -(-\frac{1}{2}h(-x) + \frac{1}{2}h(x)) = -(\frac{1}{2}h(x) - \frac{1}{2}h(-x)) = -g(x)$, we know that $g$ is odd. Also since $(f+g)(x) = f(x) + g(x) = \frac{1}{2}h(x) + \frac{1}{2}h(-x) + \frac{1}{2}h(x) - \frac{1}{2}h(-x) = h(x)$, we know that $f + g = h$. Therefore, $h = f + g \in V_e + V_o$.

All that remains is to show that $V_e + V_o$ is a direct sum. To do this, we show that $V_e \cap V_o = \{0\}$, which by theorem 1.46 means that $V_e + V_o$ is a direct sum. Let $f \in V_e \cap V_o$. Then $f$ is both even and odd. So $f(-x) = f(x)$ and $f(-x) = -f(x)$. Then $f(x) = f(-x) = -f(x)$. Add $f(x)$ to both sides to get $2f(x) = 0$. Since $2 \neq 0$, we know $f(x) = 0 = 0(x)$. Therefore, $f = 0$. So finally, we see that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

# 2.A Span and Linear Independence

## Examples

**2.18 Prove that $(2,3,1), (1,-1,2), (7,3,c)$ is linearly dependent if and only if $c = 8$**

First we show that if $c = 8$, then $(2,3,1), (1,-1,2), (7,3,c)$ is linearly dependent. Because $2(2,3,1) + 3(1,-1,2) + -1(7,3,c) = (4,6,2) + (3,-3,6) + (-7,-3,-8) = (7,3,8) + (-7,-3,-8) = 0$ and $2, 3, -1$ are not all zero, we know that $(2,3,1), (1,-1,2), (7,3,c)$ is linearly dependent.

Now we prove the converse. Assume that $(2,3,1), (1,-1,2), (7,3,c)$ is linearly dependent. We must show that $c = 8$. We know that $x(2,3,1) + y(1,-1,2) + z(7,3,c) = 0$ for some $x, y, z \in \mathbf{F}$ such that $x, y, z$ are not all zero. Then $(0,0,0) = x(2,3,1) + y(1,-1,2) + z(7,3,c) = (2x, 3x, x) + (y, -y, 2y) + (7z, 3z, cz) = (2x + y + 7z, 3x - y + 3z, x + 2y + cz)$. Therefore, we know that $2x + y + 7z = 0$, $3x - y + 3z = 0$, and $x + 2y + cz = 0$.

Sum together those first two equations to get $5x + 10z = 0$ which implies $5x = -10z$, so $x = -2z$. We use this fact to perform the following substitution: $0 = 2x + y + 7z = 2 \cdot -2z + y + 7z = -4z + y + 7z = y + 3z$. Subtract $3z$ from both sides to see that $y = -3z$. Therefore, $0 = x + 2y + cz = -2z + 2 \cdot -3z + cz = -2z - 6z + cz = -8z + cz = (c-8)z$.

~~Since x,y,z are not~~

Now we would like to divide both sides by $z$ but first we must ensure that $z \neq 0$. Assume, to the contrary, that $z=0$. Then since $x=-2z$ and $y=-3z$, we know that $x=y=z=0$. However, that contradicts our assumption that $x,y,z$ are not all zero. Therefore, it must be the case that $z \neq 0$.

Now back to our equation: $0=(c-8)z$. Divide both sides by $z$ to get $c-8=0$ which implies $c=8$.

# Exercises

**1. Find a list of four distinct vectors in $\mathbf{F}^3$ whose span equals $\{(x,y,z)\in \mathbf{F}^3 : x+y+z=0\}$.**

~~We show that (1,0,-1), (0,1,-1), (2,0,-2), (0,0,0) spans~~

Let $U=\{(x,y,z)\in \mathbf{F}^3 : x+y+z=0\}$. We show that $(1,0,-1)$, $(0,1,-1)$, $(2,0,-2)$, $(0,0,0)$ spans $U$. Let $u \in U$. We must show that there's some $a_1,\ldots,a_4 \in \mathbf{F}$ such that $a_1(1,0,-1)+a_2(0,1,-1)+a_3(2,0,-2)+a_4(0,0,0)=u$. Since $u \in U$, we know that there's some $x,y,z \in \mathbf{F}$ such that $x+y+z=0$. Add $-x-y$ to both sides to see that $z=-x-y$.

Choose $a_1=-x$, $a_2=y$, $a_3=x$, and $a_4=0$. Then observe the following:

$$
\begin{aligned}
&a_1(1,0,-1)+a_2(0,1,-1)+a_3(2,0,-2)+a_4(0,0,0)\\
&\quad = -x(1,0,-1)+y(0,1,-1)+x(2,0,-2)+0(0,0,0)\\
&\quad = (-x,0,x)+(0,y,-y)+(2x,0,-2x)+0\\
&\quad = (-x+2x,\ y,\ x-y-2x)\\
&\quad = (x,y,-x-y)\\
&\quad = (x,y,z)\\
&\quad = u
\end{aligned}
$$

**2. Prove or give a counterexample: If $v_1,v_2,v_3,v_4$ spans $V$, then the list $v_1-v_2,\ v_2-v_3,\ v_3-v_4,\ v_4$ also spans $V$.**

Let $A$ be the list $v_1,v_2,v_3,v_4$ and $B$ be the list $v_1-v_2, v_2-v_3, v_3-v_4, v_4$. Assume that $A$ spans $V$. We show that $B$ also spans $V$.

Since vector spaces are closed under addition, we know that

$v_1-v_2, v_2-v_3, v_3-v_4, v_4 \in \text{span}(v_1,v_2,v_3,v_4)$. Therefore, span(B) is a subspace of span(A).

Since $1(v_1-v_2)+1(v_2-v_3)+1(v_3-v_4)+1v_4 = v_1-v_2+v_2-v_3+v_3-v_4+v_4 = v_1$, we know that $v_1 \in \text{span}(B)$. Since $0(v_1-v_2)+1(v_2-v_3)+1(v_3-v_4)+1v_4 = v_2-v_3+v_3-v_4+v_4 = v_2$, we know that $v_2 \in \text{span}(B)$. Since $0(v_1-v_2)+0(v_2-v_3)+1(v_3-v_4)+1v_4 = v_3-v_4+v_4 = v_3$, we know that $v_3 \in \text{span}(B)$. Then because $v_1,v_2,v_3,v_4 \in \text{span}(B)$, it must be the case that span(A) is a subspace of span(B).

Since span(A) and span(B) are mutually subspaces of each other, we know that span(B) = span(A) = V. Therefore, B spans V.

3. Suppose $v_1,\dots,v_m$ is a list of vectors in V. For $k \in \{1,\dots,m\}$, let $w_k = v_1+\cdots+v_k$. Show that $\text{span}(v_1,\dots,v_m) = \text{span}(w_1,\dots,w_m)$.

Since vector spaces are closed under addition and each of the w's is a sum of v's, it should be clear that $w_1,\dots,w_m \in \text{span}(v_1,\dots,v_m)$. Therefore, $\text{span}(w_1,\dots,w_m)$ is a subspace of $\text{span}(v_1,\dots,v_m)$.

We show that $v_1,\dots,v_m \in \text{span}(w_1,\dots,w_m)$ using m steps.

Step 1: Since $w_1 = v_1$, we know $v_1 \in \text{span}(w_1,\dots,w_m)$.

Step k for $k \in \{2,\dots,m\}$: Since $-1w_{k-1}+1w_k = -(v_1+\cdots+v_{k-1}) + (v_1+\cdots+v_k) = \begin{matrix} -v_1-\cdots-v_{k-1} \\ +v_1+\cdots+v_{k-1}+v_k \end{matrix} = v_k$, we know that $v_k \in \text{span}(w_1,\dots,w_m)$.

Because $v_1,\dots,v_m \in \text{span}(w_1,\dots,w_m)$, we know that $\text{span}(v_1,\dots,v_m)$ is a subspace of $\text{span}(w_1,\dots,w_m)$. Therefore, since $\text{span}(v_1,\dots,v_m)$ and $\text{span}(w_1,\dots,w_m)$ are mutually subspaces of each other, $\text{span}(v_1,\dots,v_m) = \text{span}(w_1,\dots,w_m)$.

Linear Algebra Done Right 2.A

**4a. Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.**

First we show that if a list of length one in a vector space is linearly independent, then the vector $v$ in the list is not zero. ~~the linear combinations of the list can be written~~ Consider the linear combination $1v$. If $v=0$, then $1v=0$. Since 1 is nonzero, this would mean the list is linearly dependent. Therefore, $v \neq 0$.

Now we prove the converse. Assume that $v \neq 0$. We must show that the list is linearly independent. ~~Let av be an arbitrary linear combination of the list where a is a scalar a ∈ F~~ Let $a$ be some element of $\mathbf{F}$ such that $av=0$. By the result from Exercise 1.B.2 and the fact that $v \neq 0$, we know $a=0$. Since all linear combinations of the list ~~can be written as~~ that equal zero can be written as $av$, we know the list is linearly independent.

**4b. Show that a list of length two in vector space is linearly independent if and only if neither of the two vectors is a scalar multiple of the other.**

Assume that the list $L$ of length 2 is linearly independent. We show that neither of the two vectors are a scalar multiple of the other by contradiction. Assume that ~~m~~ $v$ and $w$ are distinct vectors in $L$ such that $w=av$ for some $a \in \mathbf{F}$. Since $L$ is linearly independent, ~~a≠0~~ $v \neq 0$ and $w \neq 0$. Therefore, $a \neq 0$. Observe $-av + 1w = -av + av = 0$. Since $-a \neq 0$ and $1 \neq 0$, we know ~~the~~ that $L$ is linearly dependent which is a contradiction.

~~Conversely, assume that~~ The converse is as follows: ~~If a list L of length 2 is~~ if neither vector in a list of length two is a scalar multiple of the other, then that list is linearly independent. We prove the contrapositive: if a list $L$ of length 2 is linearly dependent, then one of the vectors is a scalar multiple of the other. Let $v$ and $w$ be the vectors in $L$. Since $L$ is linearly dependent, we know that $a_1v + a_2w = 0$ for some $a_1, a_2 \in \mathbf{F}$ such that $a_1$ and $a_2$ are not both zero. ~~We consider these cases separately~~ Subtract $a_2w$ from both sides to get

$a_1 v = -a_2 w$. We know $a_1 \neq 0$ or $a_2 \neq 0$ so let's consider these cases separately.

Case 1: $a_1 \neq 0$. Divide both sides by $a_1$ to get $v_1 = -\frac{a_2}{a_1} w$.

Case 2: $a_2 \neq 0$. Divide both sides by $-a_2$ to get $w = -\frac{a_1}{a_2} v$.

In both cases, one vector is a scalar multiple of the other.

**5. Find a number $t$ such that $(3,1,4), (2,-3,5), (5,9,t)$ is not linearly independent in $\mathbf{R}^3$.**

Choose $t=2$. We show that $(3,1,4), (2,-3,5), (5,9,t)$ is not linearly independent in $\mathbf{R}^3$. Observe $-3(3,1,4) + 2(2,-3,5) + 1(5,9,t) = (-9,-3,-12) + (4,-6,10) + (5,9,2) = (-5,-9,-2) + (5,9,2) = (0,0,0) = 0$. Because this shows a linear combination of the list—with some nonzero coefficients—equalling 0, we know $(3,1,4), (2,-3,5), (5,9,t)$ is linearly dependent.

**Exercise 6 is just the exercise left for the reader from Example 2.18 which we already covered.**

**7a. Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $1+i, 1-i$ is linearly independent.**

Assume that for some $a, b \in \mathbf{R}$, $a(1+i) + b(1-i) = 0$. We must show that $a=b=0$. Observe $0+0i = a(1+i) + b(1-i) = a + ai + b - bi = a + b + ai - bi = (a+b) + (a-b)i$. Therefore, $a+b=0$ and $a-b=0$. Add these equations to get $a=0$. Then $b = 0+b = a+b = 0$, so $a=b=0$.

**7b. Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $1+i, 1-i$ is linearly dependent.**

Observe $i(1+i) + 1(1-i) = i + -1 + 1 - i = 0$. Since $i$ and $1$ are not 0, $1+i, 1-i$ is linearly dependent.

Linear Algebra Done Right 2.A

8. Suppose $v_1, v_2, v_3, v_4$ is linearly independent in $V$. Prove that the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is also linearly independent.

Let $A = v_1, v_2, v_3, v_4$ and $B = v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$. By Exercise 2.A.2, we know $\text{span}(A) = \text{span}(B)$. Now assume, for the purpose of contradicting, that $B$ is linearly independent. Then by Theorem 2.19, we can remove one element of list $B$ to create a list $C$ of length 3 ~~which spans~~ ~~spans~~ such that $\text{span}(C) = \text{span}(B)$. So $\text{span}(C) = \text{span}(A)$. Since $C$ spans $\text{span}(A)$, by Theorem 2.22 we know the length of $A$ is less than or equal to the length of $C$. In other words, $4 \leq 3$ which is a contradiction.

9. Prove or give a counterexample: If $A = v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in $V$, then $B = 5v_1 - 4v_2, v_2, v_3, \ldots, v_m$ is linearly independent.

We prove the statement by contradiction. Assume that $A$ is linearly independent but $B$ is not.

Recall that the span of a list of vectors is the smallest subspace that includes every vector in the list. Also recall that vector spaces are closed under addition and scalar multiplication. Therefore, $5v_1 - 4v_2 \in \text{span}(A)$. Of course $v_2, \ldots, v_m \in \text{span}(A)$. So $\text{span}(B)$ must be a subspace of $\text{span}(A)$.

Similarly, $v_1 = v_1 - \frac{4}{5}v_2 + \frac{4}{5}v_2 = \frac{1}{5}\cdot\frac{5}{1}v_1 - \frac{1}{5}\cdot\frac{4}{1}v_2 + \frac{4}{5}v_2 = \frac{1}{5}(5v_1 - 4v_2) + \frac{4}{5}v_2 \in \text{span}(B)$. Also, $v_2, \ldots, v_m \in \text{span}(B)$. Therefore, $\text{span}(A)$ must be a subspace of $\text{span}(B)$. Since $\text{span}(A)$ and $\text{span}(B)$ are mutually subspaces of each other, $\text{span}(A) = \text{span}(B)$. For convenience, let $U = \text{span}(A) = \text{span}(B)$.

Because $B$ is linearly dependent, we can remove some element from it to make a list $C$ of length $m-1$ and with the same span, $U$. Since $A$ is linearly independent in $U$ and $C$ spans $U$, we know by Theorem 2.22 that the length of $A$ is less than or equal to the length of $C$. That is, $m \leq m-1$ which is our desired contradiction.

Linear Algebra Done Right 2.A

11. Prove or give a counterexample: If $v_1, \dots, v_m$ and $w_1, \dots, w_m$ are linearly independent lists of vectors in $V$, then the list $v_1 + w_1, \dots, v_m + w_m$ is linearly independent

We show a counterexample. Think of the vector space $\mathbf{R}$ over $\mathbf{R}$. Then the length-1 lists 1 and -1 are each linearly independent. However, $1 - 1 = 0$ is not.

12. Suppose $v_1, \dots, v_m$ is linearly independent in $V$ and $w \in V$. Prove that if $v_1 + w, \dots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \dots, v_m)$.

By Theorem 2.19 we know that there's some $k \in \{1, 2, \dots, m\}$ such that $v_k + w \in \operatorname{span}(v_1 + w, \dots, v_{k-1} + w)$. Then we can write for some $a_1, \dots, a_{k-1} \in \mathbf{F}$

$$v_k + w = a_1(v_1 + w) + \dots + a_{k-1}(v_{k-1} + w) = \sum_{i=1}^{k-1} a_i(v_i + w) = \sum_{i=1}^{k-1} (a_i v_i + a_i w)$$

$$= \sum_{i=1}^{k-1} a_i v_i + \sum_{i=1}^{k-1} a_i w = \sum_{i=1}^{k-1} a_i v_i + w \sum_{i=1}^{k-1} a_i$$

Let $x = \sum_{i=1}^{k-1} a_i v_i$ and $y = \sum_{i=1}^{k-1} a_i$.

Then $v_k + w = x + wy$. ~~Subtract wy from both sides~~ Add $-v_k - wy$ to both sides to get $w - wy = x - v_k$. So $(1-y)w = x - v_k$. ~~Divide both sides~~

We show that $1 - y \neq 0$. Assume to the contrary that $1 - y = 0$. Then $0 = 0w = (1-y)w = x - v_k = (a_1 v_1 + \dots + a_{k-1} v_{k-1}) + -1 v_k$. Notice how the rightmost side is a linear combination of $v_1, \dots, v_m$ which equals 0. Since $-1 \neq 0$, this contradicts $v_1, \dots, v_m$ being linearly independent.

Now we revisit $(1-y)w = x - v_k$. Multiply both sides by $\frac{1}{(1-y)}$ to get $w = \frac{1}{1-y}(x - v_k) = \frac{1}{1-y}(a_1 v_1 + \dots + a_{k-1} v_{k-1} - v_k) \in \operatorname{span}(v_1, \dots, v_m)$.

13. Suppose $v_1, \dots, v_m$ is linearly independent in $V$ and $w \in V$. Show that $v_1, \dots, v_m, w$ is linearly independent if and only if $w \notin \operatorname{span}(v_1, \dots, v_m)$.

First we show that if $v_1, \dots, v_m, w$ is linearly independent, then $w \notin \operatorname{span}(v_1, \dots, v_m)$. Assume, for the purpose of contradicting, that

$w \in \text{span}(v_1, \ldots, v_m)$. Then there's some $a_1, \ldots, a_m \in \mathbf{F}$ such that $w = a_1v_1 + \cdots + a_mv_m$. So $0 = -w + w = -(a_1v_1 + \cdots + a_mv_m) + w =$ $-a_1v_1 - \cdots - a_mv_m + 1w$. However, since $1 \neq 0$, this contradicts $v_1, \ldots, v_m, w$ being linearly independent.

Now we show the converse. Assume that $w \notin \text{span}(v_1, \ldots, v_m)$. We must show that $v_1, \ldots, v_m, w$ is linearly independent. Let $a_1, \ldots, a_{m+1}$ be a list of scalars such that $a_1v_1 + \cdots + a_mv_m + a_{m+1}w = 0$. Solve for $a_{m+1}w$ to get $a_{m+1}w = -a_1v_1 - \cdots - a_mv_m$.

We show that $a_{m+1} = 0$. Assume, to the contrary, that $a_{m+1} \neq 0$. Then multiply both sides of the equation by $\frac{1}{a_{m+1}}$ to get $w = \frac{1}{a_{m+1}}(-a_1v_1 - \cdots - a_mv_m) \in \text{span}(v_1, \ldots, v_m)$ which contradicts our main assumption.

Therefore, $0 = a_1v_1 + \cdots + a_mv_m + a_{m+1}w = a_1v_1 + \cdots + a_mv_m$. Since $v_1, \ldots, v_m$ is linearly independent, $a_1 = \cdots = a_m = 0 = a_{m+1}$. Therefore, $v_1, \ldots, v_m, w$ is linearly independent.

**14.** Suppose $v_1, \ldots, v_m$ is a list of vectors in $V$. For $k \in \{1, \ldots, m\}$, let $w_k = v_1 + \cdots + v_k$. Show that the list $A = v_1, \ldots, v_m$ is linearly independent if and only if the list $B = w_1, \ldots, w_m$ is linearly independent.

Lemma 1: Assume there are two lists of vectors with the same length and the same span. If one of them is linearly independent, then the other one is too.

Let $X$ and $Y$ be lists of vectors with length $n$ and span $V$. Assume that $X$ is linearly independent. Assume, for the purpose of contradicting, that $Y$ is linearly dependent. Then by the linear dependence lemma (2.19), there's some list $Z$ of length $n-1$ with the same span as $Y$. Since $X$ is linearly independent in $V$ and $Z$ spans $V$, Theorem 2.22 says that the length of $X$ is less than or equal to the length of $Z$. That is $n \leq n-1$, which is a contradiction. Therefore, $Y$ is linearly independent.

Now we show that $A$ and $B$ have the same span. Since $w_k = v_1 + \cdots + v_k$, we know that every element of the list $B$ is a linear combination of A. Therefore, span(B) is a subspace of span(A).

Similarly, notice that $w_k - w_{k-1} = \sum_{i=1}^{k} v_i - \sum_{i=1}^{k-1} v_i = v_k + \sum_{i=1}^{k-1} v_i - \sum_{i=1}^{k-1} v_i = v_k$.

Therefore, every element of A is a linear combination of B. So span(A) is a subspace of span(B). Because span(A) and span(B) are mutually subspaces of each other, span(A) = span(B).

Then since A and B have the same length m and the same span, we can apply Lemma 1 to see that A is linearly independent if and only if B is linearly independent.

15. Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

Fix x as an arbitrary element of $\mathbf{F}$.

Assume, to the contrary, that there is such a list of six polynomials L. ~~Let~~ For every $k \in \{0, 1, \dots, 4\}$ let $f_k$ be the function defined by $f(x) = x^k$. We show that $f_0, f_1, \dots, f_4$ spans $\mathcal{P}_4(\mathbf{F})$.

Let g be some polynomial of degree 4. Then there's some $a_0, a_1, \dots, a_4 \in \mathbf{F}$ such that $g(x) = a_0 + a_1 x + \cdots + a_4 x = a_0 f_0(x) + a_1 f_1(x) + \cdots + a_4 f_4(x) = (a_0 f_0)(x) + (a_1 f_1)(x) + \cdots + (a_4 f_4)(x) = (a_0 f_0 + a_1 f_1 + \cdots + a_4 f_4)(x)$. So g is a linear combination of $f_0, f_1, \dots, f_4$.

Let $h \in \text{span}(f_0, f_1, \dots, f_4)$. Then $h(x) = (b_0 f_0 + b_1 f_1 + \cdots + b_4 f_4)(x) = (b_0 f_0)(x) + (b_1 f_1)(x) + \cdots + (b_4 f_4)(x) = b_0 f_0(x) + b_1 f_1(x) + \cdots + b_4 f_4(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_4 x^4$. Therefore, $h \in \mathcal{P}_4(\mathbf{F})$.

So by mutual inclusion, $\mathcal{P}_4(\mathbf{F}) = \text{span}(f_0, f_1, \dots, f_4)$. Since $f_0, f_1, \dots, f_4$ spans $\mathcal{P}_4(\mathbf{F})$ and L is linearly dependent in $\mathcal{P}_4(\mathbf{F})$, Theorem 2.22 tells us that the length of L is less than or equal to the length of $f_0, f_1, \dots, f_4$. That is $6 \leq 5$, which is a contradiction.

**16. Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.**

Fix $x$ as an arbitrary element of $\mathbf{F}$.
For every integer $0 \le k \le 4$, let $f_k : \mathbf{F} \to \mathbf{F}$ be defined $f_k(x) = x^k$. In Exercise 2.A.15, we saw that $f_0, f_1, \ldots, f_4 \in \mathcal{P}_4(\mathbf{F})$ (indeed it spans $\mathcal{P}_4(\mathbf{F})$). Additionally, we show that $f_0, f_1, \ldots, f_4$ is linearly independent.

~~Let h be a linear combination of $f_0, f_1, \ldots, f_4$. Then for some $a_0, a_1, \ldots, a_4 \in \mathbf{F}$, h 0 =~~

Let $a_0, a_1, \ldots, a_4$ be elements of $\mathbf{F}$ such that $0 = 0(x) = (a_0f_0 + a_1f_1 + \cdots + a_4f_4)(x) = (a_0f_0)(x) + (a_1f_1)(x) + \cdots + (a_4f_4)(x) = a_0f_0(x) + a_1f_1(x) + \cdots + a_4f_4(x) = a_0 + a_1x + a_2x^2 + \cdots + a_4x^4$. Because the coefficients of a polynomial are uniquely determined, we know that $a_0 = a_1 = \cdots = a_4 = 0$. Therefore, $f_0, f_1, \ldots, f_4$ is linearly independent.

~~Let L be~~ Let $L$ be a list that spans $\mathcal{P}_4(\mathbf{F})$. Since $f_0, f_1, \ldots, f_4$ is a length-5 linearly independent list in $\mathcal{P}_4(\mathbf{F})$, Theorem 2.22 tells us that ~~5 ≤ the leng~~ 5 is less than or equal to the length of $L$. That is, the length of $L$ is greater than 4. Therefore, no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

**17. Prove that $V$ is infinite dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors $V$ such that $v_1, \ldots, v_m$ is linearly independent for every positive integer $m$.**

First, we show that if there's some sequence of vectors $v_1, v_2, \ldots$ in $V$ such that $v_1, \ldots, v_m$ is linearly independent for every integer $m$, then $V$ is infinite dimensional. Assume that such a sequence $v_1, v_2, \ldots$ exists. Let $L$ be a list of vectors in $V$ and let $n$ be the length of $L$. We know $v_1, \ldots, v_{n+1}$ is linearly independent. Then by Theorem 2.22, the length of $v_1, \ldots, v_{n+1}$ is less ~~that~~ than or equal to the length of every list that spans $V$. Since $n+1 > n$, we know that $L$ does not span $V$. Therefore because no list of vectors spans $V$, we know $V$ is infinite dimensional.

Now we prove the converse. Assume that $V$ is infinitely dimensional. We show that there's some sequence $v_1, v_2, \ldots$ ~~in V s~~ of vectors in $V$ such that $v_1, \ldots, v_m$ is linearly independent for ~~all~~ every positive integer $m$. We prove this via induction. If $V$ were $\{0\}$, then the empty list would span $V$. So since no list of vectors spans $V$, $V \ne \{0\}$. Therefore, there must be some ~~a~~ $v_1 \in V$ such that $v_1 \notin$ ~~$\{0\}$~~ span(()). Therefore the length-1 list $v_1$ is linearly independent.

For ~~every~~ some positive integer $k$, assume that there's some ~~linear~~ list $v_1, \dots, v_k$ such that $v_1, \dots, v_k$ is linearly independent. Since no list of vectors spans $V$, $v_1, \dots, v_k$ does not span $V$. Then there's some $v_{k+1} \in V$ such that $v_{k+1} \notin v_1, \dots, v_k$. Therefore, $v_1, \dots, v_{k+1}$ is linearly independent.

Therefore there's some infinite sequence $v_1, v_2, \dots$ of vectors in $V$ such that for every positive integer $m$, $v_1, \dots, v_m$ is linearly independent.

**18. Prove that $\mathbf{F}^\infty$ is ~~infinitely~~ infinite-dimensional.**

Let $v_1, v_2, \dots$ be the sequence $(1,0,0,0,\dots), (0,1,0,0,\dots), (0,0,1,0,\dots), \dots$ We show that for every positive integer $m$, $v_1, \dots, v_m$ is linearly independent. Let $a_1, \dots, a_m$ be scalars such that $a_1v_1 + \dots + a_mv_m = 0$. Then $0 = (0,0,\dots,0) = a_1v_1 + \dots + a_mv_m = a_1(1,0,0,\dots,0,0) + a_2(0,1,0,\dots,0,0) + \dots + a_m(0,0,0,\dots,0,0) = (a_1,0,0,\dots,0,0) + (0,a_2,0,\dots,0,0) + \dots + (0,0,0,\dots,0,a_m) = (a_1, a_2, \dots, a_m)$. So $a_1 = \dots = a_m = 0$. Therefore, $v_1, \dots, v_m$ is linearly independent. It the follows from~~e~~ Exercise 17 that $\mathbf{F}^\infty$ is infinite-dimensional.

**19. Prove that the real vector space of all continuous real-valued functions on the interval $[0,1]$ is infinite dimensional**

Fix $x$ as an arbitrary real number in the interval $[0,1]$.
For every positive integer $i$, let $f_i : [0,1] \to \mathbf{R}$ be defined by

$$f_i(x) = \begin{cases} x - \frac{1}{i} & \text{if } x \geq \frac{1}{i} \\ 0 & \text{if } x \leq \frac{1}{i} \end{cases}$$

Then $f_i$ is continuous on the interval $[0,1]$.

Since $1 \geq \frac{1}{2}$, $f_2(1) = 1 - \frac{1}{2} = \frac{1}{2} \neq 0$. Therefore, $f_2 \notin \operatorname{span}()$. Let there be an arbitrary integer $k \geq 2$. We show that $f_{k+1} \notin \operatorname{span}(f_2, \dots, f_k)$. Assume to the contrary that $f_{k+1} \in \operatorname{span}(f_2, \dots, f_k)$. Then $f_{k+1}(x) = (a_2f_2 + \dots + a_kf_k)(x) = a_2f_2(x) + \dots + a_kf_k(x)$ for some $a_2, \dots, a_k \in \mathbf{R}$. Observe $f_{k+1}(\frac{1}{k}) = a_2f_2(\frac{1}{k}) + \dots + a_kf_k(\frac{1}{k})$.

Note that for every ~~i4~~ integer $2 \leq j \leq k$, we can take the reciprochal to see that $\frac{1}{j} \geq \frac{1}{k}$, so $\frac{1}{k} \leq \frac{1}{j}$. Therefore, $f_j(\frac{1}{k}) = 0$. Also, we know $k < k+1$. Take the reciprocal to get $\frac{1}{k} > \frac{1}{k+1}$. Therefore, we know that

$f_{k+1}(\frac{1}{k}) = \frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k^2+k} \neq 0$. Finally, our contradiction is $0 \neq f_{k+1}(\frac{1}{k}) = a_2 f_2(\frac{1}{k}) + \cdots + a_k f_k(\frac{1}{k}) = 0$.

Therefore, for every integer $m \geq 2$, $f_2, \ldots, f_m$ is linearly independent. So by Exercise 17, the vector space of all continuous functions on the interval $[0,1]$ is infinite-dimensional.

**20. Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_k(2)=0$ for each $k \in \{0, \ldots, m\}$. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.**

Assume to the contrary that $p_0, p_1, \ldots, p_m$ is linearly independent. Let $i: \mathbf{F} \to \mathbf{F}$ be the identity function. Recall that the length $m+1$ list $i^0, i^1, \ldots, i^m$ spans $\mathcal{P}_m(\mathbf{F})$. Then by Theorem 2.22, the length $m+2$ list $p_0, p_1, \ldots, p_m, i$ must be linearly dependent. Therefore, $i \in \operatorname{span}(p_0, p_1, \ldots, p_m)$.

Then for some $a_0, \ldots, a_m \in \mathbf{F}$, $2 = i(2) = (a_0 p_0 + a_1 p_1 + \cdots + a_m p_m)(2) = a_0 p_0(2) + a_1 p_1(2) + \cdots + a_m p_m(2) = 0$, which is a contradiction.

# 2.B Bases

**1. Find all vector spaces that have exactly one basis**

$\{0\}$ has exactly one basis: the empty list $()$.

No other vector space has exactly one basis. To see this is true assume ~~to the contrary~~ that $U \neq \{0\}$ is a vector space. Since $U \neq \{0\}$, there must be some nonzero $u \in U$. Then $-u \in U$ and $-u \neq 0$. By Exercise 2A.4, the length-1 lists $u$ and $-u$ are both linearly independent. Then Theorem 2.32 tells us that ~~$u, w_1, \ldots, w_n$ and $-u, v_1, \ldots, v_n$ are bases for some vectors $w_1, \ldots, w_n$ and $v_1, \ldots, v_n$~~ both those lists can be extended to bases of $U$. Since those bases would have different first elements, they are distinct. Therefore, $U$ has at least two bases.

**2a. Prove that $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is a basis of $\mathbf{F}^n$.**

Let $v \in \mathbf{F}^n$. Then for some $v_1, \ldots, v_n \in \mathbf{F}$, $v = (v_1, \ldots, v_n) =$

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$(v_1, 0, \ldots, 0) + (0, v_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, v_n) = v_1(1, 0, \ldots, 0) + v_2(0, 1, 0, \ldots, 0) + \cdots + v_n(0, \ldots, 0, 1)$. Now to ~~show~~ complete the criterion for basis, we must show that this is the only way to write $v$ in this form.

Assume that for some $a_1, \ldots, a_n \in \mathbf{F}$, $v = (v_1, \ldots, v_n) = a_1(1, 0, \ldots, 0) + a_2(0, 1, 0, \ldots, 0) + \cdots + a_n(0, \ldots, 0, a_n) = (a_1, 0, \ldots, 0) + (0, a_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, a_n) = (a_1, \ldots, a_n)$. Therefore, $v_1 = a_1, \ldots, v_n = a_n$.

**2b. Prove that $(1,2), (3,5)$ is a basis of $\mathbf{F}^2$.**

Let $v = (v_1, v_2) \in \mathbf{F}^2$. Consider the equation $v = a_1(1,2) + a_2(3,5)$. If we can show that there's exactly one $a_1, a_2 \in \mathbf{F}$ where that equation is true, we'll know that $(1,2), (3,5)$ meets the criterion for basis.

Observe $(v_1, v_2) = v = a_1(1,2) + a_2(3,5) = (a_1, 2a_1) + (3a_2, 5a_2) = (a_1 + 3a_2, 2a_1 + 5a_2)$. So we're looking for all $a_1, a_2 \in \mathbf{F}$ such that $v_1 = a_1 + 3a_2$ and $v_2 = 2a_1 + 5a_2$. Note that $2v_1 - v_2 = 2(a_1 + 3a_2) - (2a_1 + 5a_2) = 2a_1 + 6a_2 - 2a_1 - 5a_2 = a_2$, so $a_2 = 2v_1 - v_2$. ~~Also $a_1 + 3a_2$~~ Then $v_1 = a_1 + 3a_2 = a_1 + 3(2v_1 - v_2)$. Subtract $3(2v_1 - v_2)$ from both sides to get $a_1 = v_1 - 3(2v_1 - v_2) = v_1 - 6v_1 + 3v_2 = -5v_1 + 3v_2$. Since $a_1 = -5v_1 + 3v_2$ and $a_2 = 2v_1 - v_2$ ~~are the show that~~ are the only solutions for $a_1$ and $a_2$, we know that $(1,2), (3,5)$ is a basis.

**2c. Prove that $(1, 2, -4), (7, -5, 6)$ is linearly independent in $\mathbf{F}^3$ but also prove that it does not span $\mathbf{F}^3$.**

Let $L = (1, 2, -4), (7, -5, 6) \in \mathbf{F}^3$.

First we show that $L$ is linearly independent. Let $a_1, a_2 \in \mathbf{F}$ such that $0 = a_1(1, 2, -4) + a_2(7, -5, 6) = (a_1, 2a_1, -4a_1) + (7a_2, -5a_2, 6a_2) = (a_1 + 7a_2, 2a_1 - 5a_2, -4a_1 + 6a_2)$. Then we know that $a_1 + 7a_2 = 0$, $2a_1 - 5a_2 = 0$, and $-4a_1 + 6a_2 = 0$. Observe $0 = 0 - 0 = 0 - 2 \cdot 0 = (2a_1 - 5a_2) - 2(a_1 + 7a_2) = 2a_1 - 5a_2 - 2a_1 - 14a_2 = -19a_2$. Divide both sides by $-19$ to get $a_2 = 0$. So $0 = a_1 + 7a_2 = a_1 + 7 \cdot 0 = a_1$. Therefore, $a_1 = a_2 = 0$.

Next, we show that $L$ does not span $\mathbf{F}^3$. We'll use the counterexample of $(1, 0, 0) \notin \operatorname{span}(L)$. Assume, with the goal of contradicting that $(1,0,0) \in \operatorname{span}(L)$. Then $(1,0,0) = a_1(1,2,-4) + a_2(7,-5,6)$ for some $a_1, a_2 \in \mathbf{F}$. Observe $(1,0,0) = a_1(1,2,-4) + a_2(7,-5,6) = (a_1, 2a_1, -4a_1) + (7a_2, -5a_2, 6a_2) = (a_1 + 7a_2, 2a_1 - 5a_2, -4a_1 + 6a_2)$. Therefore, $a_1 + 7a_2 = 1$, $2a_1 - 5a_2 = 0$, and $-4a_1 + 6a_2 = 0$. Add $-2a_1$ to both sides of that second equation to get $-5a_2 = -2a_1$. Multiply both sides by 2 to get $-10a_2 = -4a_1$. Observe $0 = -4a_1 + 6a_2 = -10a_2 + 6a_2 = -4a_2$. Divide both sides by $-4$ to see that $a_2 = 0$.

Then $0 = 2a_1 - 5a_2 = 2a_1 + -5 \cdot 0 = 2a_1$. Divide both sides by 2 to get $a_1 = 0$. Finally, $1 = a_1 + 7a_2 = 0 + 7 \cdot 0 = 0$ is our desired contradiction.

3. Let $U$ be the subspace of $\mathbf{R}^5$ defined by $U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}$

a. Find a basis of $U$

We show that ~~that~~ $L = (3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1)$ is a basis of $U$. First, we show that $L$ is linearly independent. Let $a_1, a_2, a_3 \in \mathbf{R}$ such that $0 = a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) = (3a_1, a_1, 0,0,0) + (0,0,7a_2, a_2, 0) + (0,0,0,0,a_3) = (3a_1, a_1, 7a_2, a_2, a_3)$. Therefore, $a_1 = a_2 = a_3 = 0$.

Next we show that $L$ spans $U$. By mutual inclusion we'll see that $\operatorname{span}(L) = U$. Let $v \in \operatorname{span}(L)$. Then for some $a_1, a_2, a_3 \in \mathbf{R}$, $v = a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,a_3) = (3a_1, a_1, 0,0,0) + (0,0,7a_2, a_2, 0) + (0,0,0,0,a_3) = (3a_1, a_1, 7a_2, a_2, a_3)$. Examine that expression and the definition of $U$ to see that $v \in U$. Let $u \in U$. Then for some $x_1, \ldots, x_5 \in \mathbf{R}$, such that $x_1 = 3x_2$ and $x_3 = 7x_4$, we know that $u = (x_1, \ldots, x_5) = (x_1, x_2, x_3, x_4, x_5) = (3x_2, x_2, 7x_4, x_4, x_5) = (3x_2, x_2, 0,0,0) + (0,0,7x_4, x_4, 0) + (0,0,0,0,x_5) = x_2(3,1,0,0,0) + x_4(0,0,7,1,0) + x_5(0,0,0,0,1)$. From that final expression we see that $u \in \operatorname{span}(L)$.

b. Extend the basis in (a) to a basis of $\mathbf{R}^5$.

We show that $A = (3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1), (1,0,0,0,0), (0,0,1,0,0)$ is a basis of $\mathbf{R}^5$. Hopefully it's intuitive that those last two vectors are not in the span of the previous ones because all the previous vectors either have zero in the one coordinate or have a nonzero in a zero coordinate. Therefore, since we know $L$ is linearly independent, $A$ must be as well.

Now we show that $\text{span}(A) = \mathbf{R}^5$ by mutual inclusion. Since every vector in $A$ is in $\mathbf{R}^5$, we know that every sum and scalar multiplication of those vectors is in $\mathbf{R}^5$. Therefore, $\text{span}(A)$ is a subset of $\mathbf{R}^5$.

Next we show that $\mathbf{R}^5$ is a subset of $\text{span}(A)$. Let $v \in \mathbf{R}^5$. Then for some $v_1, \ldots, v_5 \in \mathbf{R}$, $v = (v_1, v_2, v_3, v_4, v_5) = (v_1 - 3v_2 + 3v_2, v_2, v_3 - 7v_4 + 7v_4, v_4, v_5) = (3v_2, v_2, 0, 0, 0) + (0, 0, 7v_4, v_4, 0) + (0,0,0,0,v_5) + (v_1 - 3v_2, 0,0,0,0) + (0,0,v_3 - 7v_4, 0, 0) = v_2(3,1,0,0,0) + v_4(0,0,7,1,0) + v_5(0,0,0,0,1) + (v_1 - 3v_2)(1,0,0,0,0) + (v_3 - 7v_4)(0,0,1,0,0)$, which is a linear combination of $A$. Therefore, $v \in \text{span}(A)$.

c. Find a subspace $W$ of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

~~Choose $W = \text{span}(A)$. First, we show that $\mathbf{R}^5 = U + W$~~

Let $B = (1,0,0,0,0), (0,0,1,0,0)$ and choose $W = \text{span}(B)$. First, we show that $\mathbf{R}^5 = U + W$. Let $v \in U + W$. Then for some $u \in U$ and some $w \in W$, $v = u + w$. Since $U = \text{span}(L)$, we know that for some $a_1, a_2, a_3 \in \mathbf{R}$, $u = a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1)$. And since $W = \text{span}(B)$ we know that for some $b_1, b_2 \in \mathbf{R}$ $w = b_1(1,0,0,0,0) + b_2(0,0,1,0,0)$. Then $v = u + w = a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) + b_1(1,0,0,0,0) + b_2(0,0,1,0,0) \in \text{span}(A) \subseteq \mathbf{R}^5$. Additionally, we can follow that equality in reverse to see that $\mathbf{R}^5 \subseteq U + W$. Therefore, $\mathbf{R}^5 = U + W$.

Next, we show that $U + W$ is a direct sum. Theorem 1.46 says that this is true if we can show that $U \cap W = \{0\}$. Let $v \in U \cap W$. Then $v \in U$ and $v \in W$. So for some $a_1, a_2, a_3 \in \mathbf{R}$, $v = a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) = (3a_1, a_1, 0,0,0) + (0,0,7a_2, a_2, 0) + (0,0,0,0,a_3) = (3a_1, a_1, 7a_2, a_2, a_3)$. Also, for some $b_1, b_2 \in \mathbf{R}$, $v = b_1(1,0,0,0,0) + b_2(0,0,1,0,0) = (b_1,0,0,0,0) + (0,0,b_2,0,0) = (b_1, 0, b_2, 0, 0)$. Therefore, $(3a_1, a_1, 7a_2, a_2, a_3) = v = (b_1, 0, b_2, 0, 0)$. Then $a_1 = a_2 = 0$. Also, $b_1 = 3a_1$ and $b_2 = 7a_2$ which implies $b_1 = b_2 = 0$. Therefore, $v = (b_1, 0, b_2, 0, 0) = (0,0,0,0,0) = 0$.

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4. Let $U$ be the subspace of $\mathbb{C}^5$ defined by
$U=\{(z_1,z_2,z_3,z_4,z_5)\in\mathbb{C}^5 : 6z_1=z_2 \text{ and } z_3+2z_4+3z_5=0\}$.

a. Find a basis of $U$.

Let $A=(1,6,0,0,0),(0,0,-2,1,0),(0,0,-3,0,1)$. We show that $A$ is a basis of $U$.

First, we show that $A$ is linearly independent. Let $a_1,a_2,a_3\in \cancel{\mathbb{R}}\,\mathbb{C}$ such that $0=(0,0,0,0,0)=a_1(1,6,0,0,0)+a_2(0,0,-2,1,0)+a_3(0,0,-3,0,1)=(a_1,6a_1,0,0,0)+(0,0,-2a_2,a_2,0)+(0,0,-3a_3,0,a_3)=(a_1,6a_1,-2a_2-3a_3,a_2,a_3)$.
Therefore, $a_1=a_2=a_3=0$.

Next, we show that $\operatorname{span}(A)=U$ by mutual inclusion. Let $v\in\operatorname{span}(A)$. Then for some $a_1,a_2,a_3\in\mathbb{R}$, $v=(a_1,6a_1,-2a_2-3a_3,a_2,a_3)=(v_1,v_2,v_3,v_4,v_5)$. Since $6v_1=6a_1=v_2$ and $v_3+2v_4+3v_5=-2a_2-3a_3+2a_2+3a_3=0$, we know that $v\in U$.

Now for the other way around. Let $u\in U$. Then for some $z_1,\dots,z_5\in\mathbb{C}$ such that $6z_1=z_2$ and $z_3+2z_4+3z_5=0$, $u=(z_1,z_2,z_3,z_4,z_5)$. Solve $z_3+2z_4+3z_5=0$ for $z_3$ to get $z_3=-2z_4-3z_5$. Observe the following $u=(z_1,z_2,z_3,z_4,z_5)=(z_1,6z_1,-2z_4-3z_5,z_4,z_5)=(z_1,6z_1,0,0,0)+(0,0,-2z_4,z_4,0)+(0,0,-3z_5,0,z_5)=z_1(1,6,0,0,0)+z_4(0,0,-2,1,0)+z_5(0,0,-3,0,1)\in\operatorname{span}(A)$.

b. Extend the basis in (a) to a basis of $\mathbb{C}^5$.

Let $B=(1,6,0,0,0),(0,0,-2,1,0),(0,0,-3,0,1),(0,1,0,0,0),(0,0,1,0,0)$.
Hopefully it's intuitive that neither of those last two vectors are in the span of the previous ones because I won't show that here. But given that's true, then since the first three vectors (A) is linearly independent, B is linearly independent.

We show that $\operatorname{span}(B)=\mathbb{C}^5$. Since every vector in $B$ is in $\mathbb{C}^5$, it follows that every vector in $\operatorname{span}(B)$ is in $\mathbb{C}^5$. Then we just need to show the other way around. Let $v\in\mathbb{C}^5$. Then for some $v_1,\dots,v_5\in\mathbb{C}$,
$v=(v_1,v_2,v_3,v_4,v_5)=(v_1,v_2-6v_1+6v_1,v_3+2v_4+3v_5-2v_4-3v_5,v_4,v_5)$
$=(v_1,6v_1,0,0,0)+(0,0,-2v_4,v_4,0)+(0,0,-3v_5,0,v_5)+(0,v_2-6v_1,0,0,0)+(0,0,v_3+2v_4+3v_5,0,0)$
$=v_1(1,6,0,0,0)+v_4(0,0,-2,1,0)+v_5(0,0,-3,0,1)+(v_2-6v_1)(0,1,0,0,0)+(v_3+2v_4+3v_5)(0,0,1,0,0)\in\operatorname{span}(B)$.

Linear Algebra Done Right 2B

c. Find a subspace W of $\mathbb{C}^5$ such that $\mathbb{C}^5 = U \oplus W$.

Let $C = (0,1,0,0,0), (0,0,1,0,0)$ and choose $W = \text{span}(C)$. First, we show that $\mathbb{C}^5 = U + W$. Let $v \in U+W$. Then for some $u \in U$ and some $w \in W$, we know that $v = u + w$. Since $U = \text{span}(A)$, we know that for some $a_1, a_2, a_3 \in \mathbb{C}$ $u = a_1(1,6,0,0,0) + a_2(0,0,-2,1,0) + a_3(0,0,-3,0,1)$. Since $W = \text{span}(C)$, for some $b_1, b_2 \in \mathbb{C}$ $w = b_1(0,1,0,0,0) + b_2(0,0,1,0,0)$. Then $v = u + w = a_1(1,6,0,0,0) + a_2(0,0,-2,1,0) + a_3(0,0,-3,0,1) + b_1(0,1,0,0,0) + b_2(0,0,1,0,0)$, which is an element of $\text{span}(B) = \mathbb{C}^5$. Because that equality also works in reverse, we know that $\mathbb{C}^5 = U + W$.

Next, we show that $U+W$ is a direct sum. By Theorem 1.46, we need only show that $U \cap W = \{0\}$. Let $v \in U \cap W$. Then $v \in U$ and $v \in W$. For some $a_1, a_2, a_3 \in \mathbb{C}$, $v = a_1(1,6,0,0,0) + a_2(0,0,-2,1,0) + a_3(0,0,-3,0,1) = (a_1, 6a_1, 0,0,0) + (0,0,-2a_2, a_2, 0) + (0,0,-3a_3, 0, a_3) = (a_1, 6a_1, -2a_2 - 3a_3, a_2, a_3)$. Also, for some $b_1, b_2 \in \mathbb{C}$, $v = b_1(0,1,0,0,0) + b_2(0,0,1,0,0) = (0, b_1, b_2, 0, 0)$. Therefore, $(a_1, 6a_1, -2a_2 - 3a_3, a_2, a_3) = v = (0, b_1, b_2, 0, 0)$. It follows that $a_1 = a_2 = a_3 = 0$. Finally, $v = (a_1, 6a_1, -2a_2 - 3a_3, a_2, a_3) = (0,0,0,0,0) = 0$.

5. Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

By Theorem 2.25 we know U and W are finite dimensional. By Theorem 2.31, there's some basis $u_1, \ldots, u_n$ of U and some basis $w_1, \ldots, w_m$ of W. We show that $A = u_1, \ldots, u_n, w_1, \ldots, w_m$ spans V, by mutual inclusion. Since every element of A is in V, we know that every element of span(A) is in V.

Next we show inclusion the other way around. Let $v \in V$. Since $V = U + W$, there's some $u \in U$ and some $w \in W$ such that $v = u + w$. Then we know there's some $a_1, \ldots, a_n \in \mathbf{F}$ such that $u = a_1 u_1 + \cdots + a_n u_n$. Also, there's some $b_1, \ldots, b_m \in \mathbf{F}$ such that $w = b_1 w_1 + \cdots + b_m w_m$. Therefore, $v = u + w = a_1 u_1 + \cdots + a_n u_n + b_1 w_1 + \cdots + b_m w_m \in \text{span}(A)$.

Notice that every element of A is in $U \cup W$. Then we have our result by Theorem 2.30 which says that A, a spanning list of V, contains a basis of V.

Linear Algebra Done Right 2B

**6. Prove or give a counterexample: If $p_0, \ldots, p_3$ is a list in $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials $p_0, \ldots, p_3$ has degree 2, then $p_0, \ldots, p_3$ is not a basis of $\mathcal{P}_3(\mathbf{F})$.**

Fix $x$ as an arbitrary element of $\mathbf{F}$. For every integer $0 \leq i \leq 3$, let $z_i : \mathbf{F} \to \mathbf{F}$ be the function defined by $z_i(x) = x^i$. Let $h : \mathbf{F} \to \mathbf{F}$ be defined by $h(x) = x^2 + x^3$. Consider the list $A = z_0, z_1, h, z_3$. Note that $z_0$ is a degree 0 polynomial, $z_1$ is degree 1, $h$ is degree 3, and $z_3$ is degree 3. Therefore, $A$ is a list of polynomials such that none of the polynomials have degree 2. We show that $A$ is a basis of $\mathcal{P}_3(\mathbf{F})$.

Let $f \in \mathcal{P}_3(\mathbf{F})$. Then for some $a_0, \ldots, a_3 \in \mathbf{F}$, $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 = a_0 + a_1x + a_2x^2 + (a_3 - a_2 + a_2)x^3 = a_0 + a_1x + a_2x^2 + a_2x^3 + (a_3 - a_2)x^3 = a_0 + a_1x + a_2(x^2 + x^3) + (a_3 - a_2)x^3 = a_0z_0(x) + a_1z_1(x) + a_2h(x) + (a_3 - a_2)z_3(x) = (a_0z_0 + a_1z_1 + a_2h + (a_3 - a_2)z_3)(x)$. So $f = a_0z_0 + a_1z_1 + a_2h + (a_3 - a_2)z_3$. Then we know $f$ can be written as a linear combination of $A$. We now need to show that $f$ can only be written in one way in this form to show that $A$ meets the criterion for basis.

Let $b_1, \ldots, b_3 \in \mathbf{F}$ such that $f = b_0z_0 + b_1z_1 + b_2h + b_3z_3$. We must show that ~~$b_1 = a_1, b_2 = a_2, b_3 = a_3$, and~~ $b_0 = a_0$, $b_1 = a_1$, $b_2 = a_2$, and $b_3 = a_3 - a_2$. Observe $a_0 + a_1x + a_2x^2 + a_3x^3 = f(x) = (b_0z_0 + b_1z_1 + b_2h + b_3z_3)(x) = b_0z_0(x) + b_1z_1(x) + b_2h(x) + b_3z_3(x) = b_0 + b_1x + b_2(x^2 + x^3) + b_3x^3 = b_0 + b_1x + b_2x^2 + b_2x^3 + b_3x^3 = b_0 + b_1x + b_2x^3 + (b_2 + b_3)x^3$. Since polynomial coefficients are unique, we know $b_0 = a_0$, $b_1 = a_1$, $b_2 = a_2$, ~~$b_2 + b_3$~~ and $b_2 + b_3 = a_3$. Then $b_3 = a_3 - b_2 = a_3 - a_2$. Therefore, we have the desired result.

**7. Suppose $v_1, v_2, v_3, v_4$ is a basis of $V$. Prove that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also a basis of $V$.**

Let $v \in V$. We ~~must show that~~ know that $A = v_1, v_2, v_3, v_4$ is a basis of $V$. We must show that $B = v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also a basis of $V$. Since $A$ is a basis of $V$, we know there's some $a_1, \ldots, a_4 \in \mathbf{F}$ such that $v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$. We must show that there's some unique $b_1, \ldots, b_4 \in \mathbf{F}$ such that $v = b_1(v_1 + v_2) + b_2(v_2 + v_3) + b_3(v_3 + v_4) + b_4v_4$. Choose $b_1 = a_1$, $b_2 = a_2 - a_1$, $b_3 = a_3 - a_2 + a_1$, and $b_4 = a_4 - a_3 + a_2 - a_1$.

Then $b_1(v_1 + v_2) + b_2(v_2 + v_3) + b_3(v_3 + v_4) + b_4v_4 = b_1v_1 + b_1v_2 + b_2v_2 + b_2v_3 + b_3v_3 + b_3v_4$

$+ b_4v_4 = b_1v_1 + (b_1+b_2)v_2 + (b_2+b_3)v_3 + (b_3+b_4)v_4 = a_1v_1 + (a_1+a_2-a_1)v_2 + (a_2-a_1+a_3-a_2+a_1)v_3 + (a_3-a_2+a_1+a_4-a_3+a_2-a_1)v_4 = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = v.$

Now we must show that $b_1, \dots, b_4$ are unique. Let $c_1, \dots, c_4 \in \mathbf{F}$ such that $v = c_1(v_1+v_2) + c_2(v_2+v_3) + c_3(v_3+v_4) + c_4v_4$. We must show that $b_1 = c_1, \dots, b_4 = c_4$. Observe $v = c_1(v_1+v_2) + c_2(v_2+v_3) + c_3(v_3+v_4) + c_4v_4 = c_1v_1 + c_1v_2 + c_2v_2 + c_2v_3 + c_3v_3 + c_3v_4 + c_4v_4 = c_1v_1 + (c_1+c_2)v_2 + (c_2+c_3)v_3 + (c_3+c_4)v_4$. So $c_1v_1 + (c_1+c_2)v_2 + (c_2+c_3)v_3 + (c_3+c_4)v_4 = v = b_1v_1 + (b_1+b_2)v_2 + (b_2+b_3)v_3 + (b_3+b_4)v_4$. Since $A = v_1, v_2, v_3, v_4$ is a basis, we know those coefficients are unique. Therefore, $c_1 = b_1$, $c_1+c_2 = b_1+b_2$, $c_2+c_3 = b_2+b_3$, and $c_3+c_4 = b_3+b_4$. So $c_1+c_2 = b_1+b_2 = c_1+b_2$ which implies $c_2 = b_2$. Repeat a similar process with the other two equations to get $c_3 = b_3$ and $c_4 = b_4$.

**8. Prove or give a counterexample: If $v_1, v_2, v_3, v_4$ is a basis of $V$, and $U$ is a subspace of $V$ such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then $v_1, v_2$ is a basis of $U$.**

Let $V = \mathbf{F}^4$, $U = \{(x,y,z,z) \in \mathbf{F}^4 : x,y,z \in \mathbf{F}\}$, $v_1 = (1,0,0,0)$, $v_2 = (0,1,0,0)$, $v_3 = (0,0,1,0)$, and $v_4 = (0,0,0,1)$. Notice how $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$. Since $(0,0,1,1) \in U$ but $(0,0,1,1) \notin \operatorname{span}(v_1, v_2)$, we know that $v_1, v_2$ is not a basis of $U$.

**9. Suppose $v_1, \dots, v_m$ is a list of vectors in $V$. For $k \in \{1, \dots, m\}$, let $w_k = v_1 + \cdots + v_k$. Show that $v_1, \dots, v_m$ is a basis of $V$ if and only if $w_1, \dots, w_m$ is a basis of $V$.**

In exercise 2A.14, we showed that $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$. Therefore $v_1, \dots, v_m$ spans $V$ if and only if $w_1, \dots, w_m$ spans $V$. We also showed in exercise 2A.14 that for two vectors wither the same length and span, if one of them is linearly independent, the the other one is too. Therefore, $v_1, \dots, v_m$ is linearly independent if and only if $w_1, \dots, w_m$ is linearly independent. That's both properties of a basis so we know that $v_1, \dots, v_m$ is a basis of $V$ if and only if $w_1, \dots, w_m$ is a basis of $V$.

10. Suppose $U$ and $W$ are subspaces of $V$ such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of $U$ and $w_1, \dots, w_n$ is a basis of $W$. Prove that $u_1, \dots, u_m, w_1, \dots, w_n$ is a basis of $V$.

Let $A = u_1, \dots, u_m$, $B = w_1, \dots, w_n$, and $C = u_1, \dots, u_m, w_1, \dots, w_n$. We must show that $C$ is a basis of $V$. Let $v \in V$. Since $V = U \oplus W$, there's some unique pair of ~~and~~ $x \in U$ and $y \in W$ such that $v = x + y$. Since $A$ is a basis of $U$, there's some unique $a_1, \dots, a_m \in \mathbf{F}$ such that $x = a_1 u_1 + \dots + a_m u_m$. Similarly, since $B$ is a basis of $W$ there's some unique $b_1, \dots, b_n \in \mathbf{F}$ such that $y = b_1 w_1 + \dots + b_n w_n$. Therefore, $v = x + y = a_1 u_1 + \dots + a_m u_m + b_1 w_1 + \dots + b_n w_n$. Notice that this is the form we consider for $C$'s criterion for basis (Theorem 2.28). Because every $v \in V$ can be written uniquely in this form, we know that $C$ is a basis of $V$.

11. Suppose $V$ is a real vector space. Show that if $v_1, \dots, v_n$ is a basis of $V$ (as a real vector space), then $v_1, \dots, v_n$ is also a basis of the complexification $V_C$ (as a complex vector space).

Let $v \in V_C$. Then $v = u + iw$ for some $u, w \in V$. Since $u \in V$ and $v_1, \dots, v_n$ is a basis of $V$, we know there's some unique $a_1, \dots, a_n \in \mathbf{R}$ such that $u = a_1 v_1 + \dots + a_n v_n$. Similarly, there's some unique $b_1, \dots, b_n \in \mathbf{R}$ such that $w = b_1 v_1 + \dots + b_n v_n$. Then $v = u + iw = a_1 v_1 + \dots + a_n v_n + i(b_1 v_1 + \dots + b_n v_n) = a_1 v_1 + \dots + a_n v_n + b_1 i v_1 + \dots + b_n i v_n = (a_1 + b_1 i) v_1 + \dots + (a_n + b_n i) v_n$. Notice how this is the form we consider for $v_1, \dots, v_n$'s criterion for basis (Theorem 2.28). Because every $v \in V_C$ can be written uniquely in this form, we know that $v_1, \dots, v_n$ is a basis of $V_C$ (as a complex vector space).

# 2C Dimension

1. Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$.

Let $U$ be a subspace of $\mathbf{R}^2$. $\dim U$ must be ~~an~~ a nonnegative integer, because dimension is defined to be the length of a list. Theorem 2.37 tells us that $\dim U \leq \dim \mathbf{R}^2 = 2$. Then $\dim U \in \{0, 1, 2\}$. We consider these cases separately.

# Linear Algebra Done Right 2C

Case 0: Then the empty list () is a basis of $U$. So $U = \text{span}() = \{0\}$.

Case 1: There's some length-1 list $v$ that's a basis of $U$. We know $v \neq 0$ because a basis is linearly independent. Let $u \in U$. Then for some $a \in \mathbf{R}$, $u = av$. Because every element of $U$ is a linear multiple of a ~~nonnega~~ nonzero vector, we know that $U$ is a line through the origin. Should really have said $U = \text{span}(v)$ so for every $a \in \mathbf{R}$, $av \in U$. I repeat this mistake in exercise 2.

Case 2: There's some length-2 basis $v_1, v_2$ of $U$. Since the length of $v_1, v_2$ is 2 and $\dim \mathbf{R}^2 = 2$, Theorem 2.38 tells us that $v_1, v_2$ is a basis of $\mathbf{R}^2$. Therefore, $U = \text{span}(v_1, v_2) = \mathbf{R}^2$.

**2. Show that the subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, all lines in $\mathbf{R}^3$ containing the origin, all planes in $\mathbf{R}^3$ containing the origin, and $\mathbf{R}^3$.**

Let $U$ be a subspace of $\mathbf{R}^3$. Since $\dim U$ is defined to be the length of a list, $\dim U$ is a ~~nonzero~~ nonnegative integer. Theorem 2.37 tells us that $\dim U \leq \dim \mathbf{R}^3 = 3$. So $\dim U \in \{0, 1, 2, 3\}$. We consider these cases separately.

Case 0: The empty list () is a basis of $U$. Then $U = \text{span}() = \{0\}$.

Case 1: There's some length-1 list $v$ which is a basis of $U$. We know $v$ is nonzero because bases are linearly independent. Let $u \in U$. Then for some $a \in \mathbf{R}$, $u = av$. Because every element of $U$ is a linear multiple of a nonzero vector, $U$ is a line containing the origin.

Case 2: There's some length-2 basis $v_1, v_2$ of $U$. We know $v_1 \neq 0$ and $v_2 \neq 0$ because bases are linearly independent. Let $u \in U$. Then for some $a_1, a_2 \in \mathbf{R}$, $u = a_1 v_1 + a_2 v_2$. When $a_2 = 0$, $u = a_1 v_1$, so $U$ contains the line containing the origin and $v_1$. Considering elements of $U$ where $a_2 \neq 0$, we can think of this as sweeping the $v_1$ line along the $v_2$ line (the line containing the origin and $v_2$). We know these lines are not parallel, because if they were, $v_2$ would be a linear multiple of $v_1$ which would contradict $v_1, v_2$ being linearly independent. The result of sweeping one infinite line along a nonparallel infinite line is a plane. Therefore, $U$ is a plane. Also, $U$ contains the origin because 0 is an element of every vector space.

Linear Algebra Done Right 2C

Case 3: There's some length-3 basis $v_1, v_2, v_3$ of $U$. Since the length of $v_1, v_2, v_3$ is 3 and $\dim \mathbb{R}^3 = 3$, Theorem 2.38 tells us that $v_1, v_2, v_3$ is a basis of $\mathbb{R}^3$. Therefore, $U = \text{span}(v_1, v_2, v_3) = \mathbb{R}^3$.

3. Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. ~~Find a basis of U.~~

a. Find a basis of $U$.

Fix $x$ as an arbitrary element of $\mathbf{F}$.
~~Let~~ For every integer $1 \leq i \leq 4$, let $f_i : \mathbf{F} \to \mathbf{F}$ be defined by $f_i(x) = -(6^i) + x^i$. Then $f_i(6) = -(6^i) + 6^i = 0$. So $f_i \in$ ~~$\mathcal{P}_4(\mathbf{F})$~~ $U$.

We show that $f_1, \ldots, f_4$ is linearly independent in $U$. Assume that for some $a_1, \ldots, a_4 \in \mathbf{F}$, $a_1 f_1 + \cdots + a_4 f_4 = 0$. Let $a_0 = -6a_1 - 6^2 a_2 - 6^3 a_3 - 6^4 a_4$. Observe:
$0 = 0(x) = (a_1 f_1 + \cdots + a_4 f_4)(x) = a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) + a_4 f_4(x) =$
$a_1(-(6^1) + x^1) + a_2(-(6^2) + x^2) + a_3(-(6^3) + x^3) + a_4(-(6^4) + a_4) =$
$-6a_1 + a_1 x^1 \cancel{+} - 6^2 a_2 + a_2 x^2 - 6^3 a_3 + a_3 x^3 - 6^4 a_4 + a_4 x^4 =$
$(-6a_1 - 6^2 a_2 - 6^3 a_3 - 6^4 a_4) + a_1 x + a_2 x^2 + a_3 x^3 + a^4 x^4 =$
$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$.
Therefore, $a_0 = a_1 = \cdots = a_4 = 0$. So $f_1, \ldots, f_4$ is linearly independent.

Then ~~$4 \leq \dim U$~~ $4 \leq \dim U \leq \dim \mathcal{P}_4(\mathbf{F}) = 5$. But notice that for $g : \mathbf{F} \to \mathbf{F}$ defined by $g(x) = 1$, $g(6) = 1 \neq 0$. Then $g \notin U$ and $g \in \mathcal{P}_4(\mathbf{F})$. Therefore, we know that $U \neq \mathcal{P}_4(\mathbf{F})$ and our inequality becomes $4 \leq \dim U < 5$ which implies $\dim U = 4$.

Since $f_1, \ldots, f_4$ is a length-4 linearly independent list of vectors in $U$, we know that $f_1, \ldots, f_4$ is a basis of $U$.

b. Extend the basis in (a) to a basis of ~~$\mathcal{P}_4$~~ $\mathcal{P}_4(\mathbf{F})$.

Since $f_1, \ldots, f_4$ is linearly independent and $g \notin U = \text{span}(f_1, \ldots, f_4)$, $f_1, \ldots, f_4, g$ is a linearly independent length-5 list of vectors in $\mathcal{P}_4(\mathbf{F})$. Therefore, $f_1, \ldots, f_4, g$ is a basis of $\mathcal{P}_4(\mathbf{F})$.

c. Find a subspace $W$ of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Let $W = \text{span}(g)$. Since $f_1, \ldots, f_4, g$ is a basis of $\mathcal{P}_4(\mathbf{F})$, ~~every~~ we know that for every $p \in \mathcal{P}_4(\mathbf{F})$, $p = a_0 g + a_1 f_1 + \cdots + a_4 f_4$ for some unique $a_0, \ldots, a_4 \in \mathbf{F}$. Since $g$ is a basis of $W$, ~~and is~~ there's some unique $w \in W$ such that $w = a_0 g$. Since ~~an~~ $f_1, \ldots, f_4$ is a basis of $U$, there's some unique $u \in U$ such that $u = a_1 f_1 + \cdots + a_4 f_4$. Then $v = w + u$ for some unique $w \in W$ and some unique $u \in U$. Therefore, $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

5. Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$.

a. Find a basis of $U$

First, note that each of the polynomials $1, (x-2)(x-5), (x^2-2^2)(x-5), (x^3-2^3)(x-5)$ is in $U$. Let $a_1, \ldots, a_4 \in \mathbf{F}$ such that
$a_1 1 + a_2(x-2)(x-5) + a_3(x^2-2^2)(x-5) + a_4(x^3-2^3)(x-5) = 0$
Notice how, when expanded, the left side will have one $x^4$ term, $a_4 x^4$. Since the right side has no $x^4$ term, $a_4 = 0$. Then we can see that the left side will have one $x^3$ term, $a_3 x^3$. Since the right side has no $x^3$ term, $a_3 = 0$. Similarly, $a_2 = 0$ which leaves $a_1 1 + 0 + 0 + 0 = a_1 = 0$. Therefore, $1, (x-2)(x-5), (x^2-2^2)(x-5), (x^3-2^3)(x-5)$ is linearly independent.

Then $4 \leq \dim U \leq \dim \mathcal{P}_4(\mathbf{F}) = 5$. Notice that the identity function, $i$ is not in $U$ because $i(2) = 2 \neq 5 = i(5)$. Since $i \in \mathcal{P}_4(\mathbf{F})$ but $i \notin U$, we know $U \neq \mathcal{P}_4(\mathbf{F})$. Then the inequality becomes $4 \leq \dim U < 5$ which implies $\dim U = 4$. Since $1, (x-2)(x-5), (x^2-2^2)(x-5), (x^3-2^3)(x-5)$ is a length-4 linearly independent list of vectors in $U$, we know it is a basis of $U$.

b. Extend the basis in (a) to a basis in $\mathcal{P}_4(\mathbf{F})$.

Similar to 3a and 3b, we can extend the basis in a with the identity function to make a basis of $\mathcal{P}_4(\mathbf{F})$.

c. Find a subspace $W$ of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

Similar to 3c.

Linear Algebra Done Right 2C

**8. Suppose $v_1, \ldots, v_m$ is linearly independent in $V$ and $w \in V$. Prove that $\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m-1$.**

Notice how for every integer $2 \leq k \leq m$, $v_1 - v_k = v_1 + w - v_k - w = (v_1+w) - (v_k - w)$. Therefore $v_1 - v_2, \ldots, v_1 - v_m \in \operatorname{span}(v_1 + w, \ldots, v_m + w)$ which means that $\operatorname{span}(v_1 - v_2, \ldots, v_1 - v_m)$ is a subspace of $\operatorname{span}(v_1 + w, \ldots, v_m + w)$.

We show that $v_1 - v_2, \ldots, v_1 - v_m$ is linearly independent. Let $a_2, \ldots, a_m \in \mathbf{F}$ such that $a_2(v_1 - v_2) + \cdots + a_m(v_1 - v_m) = 0$. Then $0 = a_2(v_1 - v_2) + \cdots + a_m(v_1 - v_m) = a_2 v_1 - a_2 v_2 + \cdots + a_m v_1 - a_m v_m = a_2 v_1 + \cdots + a_m v_1 - a_2 v_2 - \cdots - a_m v_m = (a_2 + \cdots + a_m) v_1 - a_2 v_2 - \cdots - a_m v_m$. Since $v_1, \ldots, v_m$ is linearly independent, $a_2 = \cdots = a_m = 0$. Therefore, $v_1 - v_2, \ldots, v_1 - v_m$ is linearly independent.

Because $v_1 - v_2, \ldots, v_1 - v_m$ is a linearly independent list of length $m-1$, $\dim \operatorname{span}(v_1 - v_2, \ldots, v_1 - v_m) = m - 1$. Since $\operatorname{span}(v_1 - v_2, \ldots, v_1 - v_m)$ is a subspace of $\operatorname{span}(v_1 + w, \ldots, v_m + w)$, ~~span~~ $\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq \operatorname{span}(v_1 - v_2, \ldots, v_1 - v_m) = m - 1$, by Theorem 2.37.

**9. Suppose $m$ is a positive integer and $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each $p_k$ has degree $k$. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.**

~~Since~~ Let $z$ be the identity function on $\mathbf{F}$. Then $z^0, z^1, \ldots, z^m$ is the standard basis of $\mathcal{P}_m(\mathbf{F})$. Since $z^0, z^1, \ldots, z^m$ is of length $m+1$, $\dim \mathcal{P}_m(\mathbf{F}) = m+1$. Because $p_0, p_1, \ldots, p_m$ is the right length for a basis, we need only show that it's linearly independent.

Note that $\deg p_0 = 0$. Therefore $p_0 \neq 0$ (which has degree $-\infty$). Then the list $p_0$ is linearly independent. For every integer $1 \leq k \leq m-1$ assume that $p_0, p_1, \ldots, p_k$ is linearly independent. We show that $p_0, p_1, \ldots, p_k, p_{k+1}$ is linearly independent. Since $p_{k+1}$ has a greater degree than all of $p_0, p_1, \ldots, p_k$, we know that $p_{k+1} \notin \operatorname{span}(p_0, p_1, \ldots, p_k)$. Then by Exercise 2A.13, $p_0, p_1, \ldots, p_k, p_{k+1}$ is linearly independent. So by induction $p_0, p_1, \ldots, p_m$ is linearly independent.

**10. Suppose $m$ is a positive integer. For every integer $0 \leq k \leq m$, let $p_k(x) = x^k(1-x)^{m-k}$. Show that $p_0, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.**

To denote these functions with more specific values for $m$ we write $p_{k,m}$. Fix $x$ as an arbitrary element of $\mathbf{F}$.

Linear Algebra Done Right 2C

First we show that $p_{0,1}, p_{1,1}$ is linearly independent in $\mathcal{P}_1(\mathbf{F})$. Let $a_0, a_1$ be elements of $\mathbf{F}$ such that $0 = a_0 p_{0,1}(x) + a_1 p_{1,1}(x) = a_0 x^0 (1-x)^{1-0} + a_1 x^1 (1-x)^{1-1} = a_0(1-x) + a_1 x$. Plug in $0$ for $x$ to see that $0 = a_0(1-0) + a_1 \cdot 0 = a_0$. Plug in $1$ for $x$ to see that $0 = a_0(1-1) + a_1 \cdot 1 = a_1$. Since $a_0 = a_1 = 0$, $p_{0,1}, p_{1,1}$ is linearly independent.

For every integer $k \geq 1$, assume that $p_{0,k}, \dots, p_{k,k}$ is linearly independent in $\mathcal{P}_k(\mathbf{F})$. We show that $p_{0,k+1}, \dots, p_{k+1,k+1}$ is linearly independent. Let $a_0, \dots, a_{k+1}$ be elements of $\mathbf{F}$ such that $0 = a_0 p_{0,k+1}(x) + a_1 p_{1,k+1}(x) + \dots + a_{k+1} p_{k+1,k+1}(x) =$
$a_0 x^0 (1-x)^{k+1-0} + a_1 x^1 (1-x)^{k+1-1} + \dots + a_{k+1} x^{k+1} (1-x)^{k+1-(k+1)} =$
$a_0(1-x)^{k+1} + a_1 x (1-x)^{k} + \dots + a_{k+1} x^{k+1} = a_0(1-x)^{k+1} + x(a_1(1-x)^k + \dots + a_{k+1} x^k) =$
$a_0(1-x)^{k+1} + x(a_1 x^0 (1-x)^{k-0} + \dots + a_{k+1} x^k (1-x)^{k-k}) = a_0(1-x)^{k+1} + x(a_1 p_{0,k}(x) + \dots + a_{k+1} p_{k,k}(x))$.

Plug in $0$ for $x$ to get $0 = a_0(1-0)^{k+1} + 0(a_1 p_{0,k}(x) + \dots + a_{k+1} p_{k,k}(x)) = a_0$.

Then notice that $0 = a_0(1-x)^{k+1} + x(a_1 p_{0,k}(x) + \dots + a_{k+1} p_{k,k}(x)) = x(a_1 p_{0,k}(x) + \dots + a_{k+1} p_{k,k}(x))$. Divide both sides by $x$ (we'll deal with $x=0$ next) to see that $0 = a_1 p_{0,k}(x) + \dots + a_{k+1} p_{k,k}(x)$.

For $x=0$ consider that there's no polynomial $p$ such that $p(x)=0$ for all nonzero values of $x$ but $p(0) \neq 0$, because polynomials are continuous. Therefore if $(a_1 p_{0,k} + \dots + a_{k+1} p_{k,k})(x) = 0$ for all nonzero values of $x$, it must also be zero at $x=0$.

Then since $p_{0,k}, \dots, p_{k,k}$ is linearly independent, $a_1 = \dots = a_{k+1} = 0$. Recall that $a_0$ is also $0$. Then $p_{0,k+1}, \dots, p_{k+1,k+1}$ is linearly independent. Therefore, by induction, $p_{0,m}, \dots, p_{m,m}$ is linearly independent. Because $p_{0,m}, \dots, p_{m,m}$ is of length $m+1$ and $\dim \mathcal{P}_m(\mathbf{F}) = m+1$, theorem 2.38 tells us that $p_{0,m}, \dots, p_{m,m}$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

**11. Suppose $U$ and $W$ are both four-dimensional subspaces of $\mathbf{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.**

Recall that the sum of two subspaces is the smallest subspace that contains both of them. Therefore, since $U$ and $W$ are both subspaces of $\mathbf{C}^6$, we know that $U+W$ is a subspace of $\mathbf{C}^6$. Then by Theorem 2.37, $\dim(U+W) \leq \dim \mathbf{C}^6$.

Now Theorem 2.43 tells us that $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$. Solve for $\dim(U \cap W)$ to get $\dim(U\cap W) = \dim U + \dim W - \dim(U+W) \geq 4+4-\dim \mathbf{C}^6 = 8-6 = 2$. So $\dim(U \cap W) \geq 2$.

~~Let m~~ Let $m = \dim(U \cap W) \geq 2$. Then there's some basis $v_1, \dots, v_m$ of $U \cap W$. So $v_1, v_2$ is linearly independent. Therefore by ~~Theorem~~ Exercise 2A.4b, $v_1$ and $v_2$ are two vectors in $U \cap W$ such that neither vector is a scalar multiple of the other.

**12. Suppose that $U$ and $W$ are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.**

By Theorem 2.43, $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$. Solve for $\dim(U\cap W)$ to get $\dim(U \cap W) = \dim U + \dim W - \dim(U+W) = 3+5-\dim \mathbf{R}^8 = 8-8=0$. Then the empty list $()$ is a basis of $U \cap W$. So $U \cap W = \operatorname{span}() = \{0\}$. Then by Theorem 1.46, $\mathbf{R}^8 = U \oplus W$.

**13. Suppose $U$ and $W$ are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.**

Since $U$ and $W$ are both subspaces of $\mathbf{R}^9$, $U+W$ is a subspace of $\mathbf{R}^9$. By Theorem 2.37, $\dim(U+W) \leq \dim \mathbf{R}^9$. Solve Theorem 2.43 for $\dim(U \cap W)$ to get $\dim(U\cap W) = \dim U + \dim W - \dim(U+W) \geq 5+5-\dim \mathbf{R}^9 = 10-9 = 1$. So $\dim(U \cap W) \geq 1$. Then since $1 > 0 = \dim\{0\}$, we have our desired result.

**14. Suppose $V$ is a ten-dimensional vector space and $V_1, V_2, V_3$ are subspaces of $V$ with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.**

Plug $V_2$ and $V_3$ into Theorem 2.43 and solve for $\dim(V_2 \cap V_3)$ to get $\dim(V_2 \cap V_3) = \dim V_2 + \dim V_3 - \dim(V_2 + V_3)$. Since $V_2$ and $V_3$ are both subspaces of the ten-dimensional space $V$, $\dim(V_2 + V_3) \leq 10$. Then $\dim(V_2 \cap V_3) \geq 7+7-10 = 4$.

Plug $V_1$ and $(V_2 \cap V_3)$ into Theorem 2.43 and solve for $\dim(V_1 \cap V_2 \cap V_3)$ to get $\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim(V_2 \cap V_3) - \dim(V_1 + (V_2 \cap V_3))$. Since $V_1, V_2, V_3$ are all subspaces of the ten-dimensional space $V$, $\dim(V_1+(V_2\cap V_3)) \leq 10$. Therefore, $\dim(V_1\cap V_2\cap V_3) \geq 7+4-10 = 11-10 = 1$. Since $\dim(\{0\})=0$, we know that $\dim(V_1\cap V_2\cap V_3) \neq \dim(\{0\})$. So $V_1\cap V_2\cap V_3 \neq \{0\}$.

**15. Suppose $V$ is finite dimensional and $V_1, V_2, V_3$ are subspaces of $V$ with $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. Prove that $V_1\cap V_2\cap V_3 \neq \{0\}$.**

Since the dimension of a vector space is always a nonnegative integer, $2\dim V + 1 \leq \dim V_1 + \dim V_2 + \dim V_3$.

By Theorem 2.43, $\dim V_1 + \dim V_2 = \dim(V_1+V_2) + \dim(V_1\cap V_2)$. Then $2\dim V + 1 \leq \dim V_1 + \dim V_2 + \dim V_3 = \dim(V_1+V_2) + \dim(V_1\cap V_2) + \dim V_3$.

Plug ~~[illegible]~~ $(V_1\cap V_2)$ and $V_3$ into Theorem 2.43 to obtain $\dim(V_1\cap V_2) + \dim V_3 = \dim((V_1\cap V_2)+V_3) + \dim(V_1\cap V_2\cap V_3)$. Then observe $2\dim V + 1 \leq \dim(V_1+V_2) + \dim(V_1\cap V_2) + \dim V_3 = \dim(V_1+V_2) + \dim((V_1\cap V_2)+V_3) + \dim(V_1\cap V_2\cap V_3)$.

Because $V_1+V_2$ and $(V_1\cap V_2)+V_3$ are subspaces of $V$, we know that $\dim(V_1+V_2) \leq \dim V$ and $\dim((V_1\cap V_2)+V_3) \leq \dim V$ by Theorem 2.37. Therefore, $2\dim V + 1 \leq \dim(V_1+V_2) + \dim((V_1\cap V_2)+V_3) + \dim(V_1\cap V_2\cap V_3) \leq \dim V + \dim V + \dim(V_1\cap V_2\cap V_3)$. Subtract $2\dim V$ from both sides to get $1 \leq \dim(V_1\cap V_2\cap V_3)$. Since $\dim(\{0\})=0$ and $\dim(V_1\cap V_2\cap V_3) > 0$, we have the desired result of $V_1\cap V_2\cap V_3 \neq \{0\}$.

**16. Suppose $V$ is finite-dimensional and $U$ is a subspace of $V$ with $U \neq V$. Let $n = \dim V$ and $m = \dim U$. Prove that there exist $n-m$ subspaces of $V$, each of dimension $n-1$, whose intersection equals $U$.**

By Theorem 2.37, $m = \dim U \leq \dim V = n$. By Theorem 2.39 since $U \neq V$, $\dim U \neq \dim V$, so the inequality becomes $m = \dim U < \dim V = n$. Because the dimension of a ~~vector space is always~~ finite-dimensional vector space is always a nonnegative integer, $m < n$ implies $m+1 \leq n$. Subtract $m$ from both sides to get $1 \leq n-m$.

Let $u_1, \dots, u_m$ be a basis of $U$. Since it's linearly independent in $V$, Theorem 2.32 says that $u_1, \dots, u_m$ can be extended to some basis $u_1, \dots, u_m, v_1, \dots, v_{n-m}$ of $V$. For every integer $1 \leq k \leq n-m$ let $L_k = u_1, \dots, u_m, v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_{n-m}$. Then $L_k$ is a list of length $n-1$. So $\operatorname{span}(L_k)$ is a subspace of dimension $n-1$. Since every $L_k$ contains the basis $u_1, \dots, u_m$ of $U$, $U$ is a subspace of $\operatorname{span}(L_k)$. ~~Therefore $\bigcap_{i=1}^{n-m} \operatorname{span}(L_i)$~~ Therefore,

$U$ is a subspace of $\bigcap_{i=1}^{n-m} \text{span}(L_i)$.

Next, we show that $\bigcap_{i=1}^{n-m} \text{span}(L_i)$ is a subspace of $U$. Let $w \in \bigcap_{i=1}^{n-m} \text{span}(L_i)$. Then $w \in$ ~~$L_k$~~ $\text{span}(L_k)$. So $w$ can be written as a linear combination of $L_k$. To represent this linear combination, let $a_{1,k}, \ldots, a_{m,k} \in \mathbf{F}$ and $b_{1,k}, \ldots, b_{n-m,k} \in \mathbf{F}$ such that $w = a_{1,k}u_1 + \cdots + a_{m,k}u_m +$ $b_{1,k}v_1 + \cdots + b_{n-m,k}v_{n-m}$. Because there's no $v_k$ in $L_k$, we know that $b_{k,k} = 0$. ~~Observe~~ Let $j \in \mathbf{Z}$ such that $1 \le j \le n-m$.

$$
\begin{aligned}
0 = w - w &= \left(\sum_{i=1}^{m} a_{i,1}u_i + \sum_{i=1}^{n-m} b_{i,1}v_i\right) - \left(\sum_{i=1}^{m} a_{i,j}u_i + \sum_{i=1}^{n-m} b_{i,j}v_i\right) \\
&= \sum_{i=1}^{m} (a_{i,1}u_i - a_{i,j}u_i) + \sum_{i=1}^{n-m} (b_{i,1}v_i - b_{i,j}v_i) \\
&= \sum_{i=1}^{m} (a_{i,1} - a_{i,j})u_i + \sum_{i=1}^{n-m} (b_{i,1} - b_{i,j})v_i
\end{aligned}
$$

(Turns out I could've just continued to use $k$ instead of introducing $j$. View them as interchangeable.

Notice how this is a linear combination of $u_1, \ldots, u_m, v_1, \ldots, v_{n-m}$, a linearly independent list, equalling zero. Therefore, all the coefficients are zero. Then for every $k$ and $j$, $b_{k,1} - b_{k,j} = 0$. Since $b_{k,k} = 0$, we know that $0 = b_{k,1} - b_{k,k} = b_{k,1} - 0 = b_{k,1}$. So $b_{k,1} = 0$ for every $k$.

Therefore, $w = a_{1,1}u_1 + \cdots + a_{m,1}u_m + b_{1,1}v_1 + \cdots + b_{n-m,1}v_{n-m} = a_{1,1}u_1 + \cdots + a_{m,1}u_m + 0v_1 + \cdots + 0v_{n-m} = a_{1,1}u_1 + \cdots + a_{m,1}u_m \in U$. Then $w \in U$ implies that $\bigcap_{i=1}^{n-m} \text{span}(L_i)$ is a subspace of $U$. Because we established earlier that the reverse is also true, $\bigcap_{i=1}^{n-m} \text{span}(L_i) = U$.

**17. Suppose that $V_1, \ldots, V_m$ are finite-dimensional subspaces of $V$. Prove that $V_1 + \cdots + V_m$ is finite dimensional and $\dim(V_1 + \cdots + V_m) \le \dim V_1 + \cdots + \dim V_m$.**

Lemma 1: if $U$ and $W$ are finite-dimensional subspaces of $V$, then $U+W$ is finite-dimensional. By theorem 2.31, there's some bases $u_1, \ldots, u_m$ of $U$ and $w_1, \ldots, w_n$ of $W$. Let $L = u_1, \ldots, u_m, w_1, \ldots, w_n$. We show that $L$ spans $U+W$.

Let $v \in U+W$. Then $v = u+w$ for some $u \in U$ and $w \in W$. So for some $a_1, \ldots, a_m \in \mathbf{F}$ and some $b_1, \ldots, b_n \in \mathbf{F}$, $v = u + w =$ $a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n$, which is a linear combination of $L$. Therefore, $U+W$ is a subspace of $\text{span}(L)$.

Let $v \in \text{span}(L)$. Then for some $a_1, \ldots, a_m \in \mathbf{F}$ and $b_1, \ldots, b_n \in \mathbf{F}$, $v = a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n = (a_1u_1 + \cdots + a_mu_m) + (b_1w_1 + \cdots + b_nw_n)$. Since the parentheses on the left contain a vector in $U$ and the ones on the right contain a vector in $W$, $v \in U+W$. So $\text{span}(L)$ is a subspace of $U+W$.

Because span($L$) and $U+W$ are mutually subspaces of each other, $L$ spans $U+W$. Then $U+W$ is finite-dimensional by definition.

We now use lemma 1 to show that $V_1+\cdots+V_m$ is finite-dimensional inductively. We know that $V_1,\dots,V_m$ are all individually finite-dimensional. For every integer $1\le k<m$, assume that $V_1+\cdots+V_k$ is finite-dimensional. Then by lemma 1 $(V_1+\cdots+V_k)+V_{k+1}=V_1+\cdots+V_{k+1}$ is finite-dimensional.

Finally, we show that $\dim(V_1+\cdots+V_m)\le \dim V_1+\cdots+\dim V_m$ inductively. By Theorem 2.43, $\dim(V_1+\cdots+V_m)=\dim((V_1+\cdots+V_{m-1})+V_m)=\dim(V_1+\cdots+V_{m-1})+\dim V_m - \dim((V_1+\cdots+V_{m-1})\cap V_m)\le \dim(V_1+\cdots+V_{m-1})+\dim V_m$.

For every integer $1\le k<m$, assume that $\dim(V_1+\cdots+V_m)\le\dim(V_1+\cdots+V_{m-k})+\dim V_{m-k+1}+\cdots+\dim V_m$. By Theorem 2.43 $\dim(V_1+\cdots+V_{m-k})=\dim((V_1+\cdots+V_{m-(k+1)})+V_{m-k})=\dim(V_1+\cdots+V_{m-(k+1)})+\dim V_{m-k}-\dim((V_1+\cdots+V_{m-(k+1)})\cap V_{m-k})\le\dim(V_1+\cdots+V_{m-(k+1)})+\dim V_{m-k}$. Therefore, $\dim(V_1+\cdots+V_m)\le\dim(V_1+\cdots+V_{m-k})+\dim V_{m-k+1}+\cdots+\dim V_m\le$ $\dim(V_1+\cdots+V_{m-(k+1)})+\dim V_{m-k}+\dim V_{m-k+1}+\cdots+\dim V_m=$ $\dim(V_1+\cdots+V_{m-(k+1)})+\dim V_{m-k}+\cdots+\dim V_m=\dim(V_1+\cdots+V_{m-(k+1)})+\dim V_{m-(k+1)+1}+\cdots+\dim V_m$. Therefore, we have the desired result via induction.

**18. Suppose $V$ is finite-dimensional, with $\dim V=n\ge1$. Prove that there exist one-dimensional subspaces $V_1,\dots,V_n$ of $V$ such that $V=V_1\oplus\cdots\oplus V_n$.**

Theorem 2.31 tells us there's some basis $v_1,\dots,v_n$ of $V$. Observe

$$\operatorname{span}(v_1)+\cdots+\operatorname{span}(v_n)=\{u_1+\cdots+u_n : u_1\in\operatorname{span}(v_1),\dots,u_n\in\operatorname{span}(v_n)\}$$
$$=\{a_1v_1+\cdots+a_nv_n : a_1,\dots,a_n\in\mathbf{F}\}=\operatorname{span}(v_1,\dots,v_n)=V.$$

Now we need only show that $\operatorname{span}(v_1)+\cdots+\operatorname{span}(v_n)$ is a direct sum. Let $u_1\in\operatorname{span}(v_1),\dots,u_n\in\operatorname{span}(v_n)$ such that $0=u_1+\cdots+u_n$. Then for some $a_1,\dots,a_n\in\mathbf{F}$ $0=u_1+\cdots+u_n=a_1v_1+\cdots+a_nv_n$. Since $v_1,\dots,v_n$ is linearly independent, $a_1=\cdots=a_n=0$. Therefore, $u_1=\cdots=u_n=0$. So $V=\operatorname{span}(v_1)\oplus\cdots\oplus\operatorname{span}(v_n)$.

**19. Explain why you might guess, motivated by analogy with the formula for the number of elements in the union of three finite sets, that if $V_1,V_2,V_3$ are subspaces of a finite-dimensional vector space, then $\dim(V_1+V_2+V_3)=\dim V_1+\dim V_2+\dim V_3-\dim(V_1\cap V_2)-\dim(V_1\cap V_3)-\dim(V_2\cap V_3)+\dim(V_1\cap V_2\cap V_3)$. Then either prove the formula above or give a counterexample.**

We've seen in prior scenarios there's a similarity between the cardinality of a finite set and the dimension of a finite-dimensional vector space. In this analogy, a union of two finite sets is similar to the sum of two subspaces. Set intersection works on subspaces, so there's no need for a different, analogous operation.

The formula for the number of elements in a union of three finite sets is $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Notice how if you swap out the operations as described in the analogy above, you get the formula for the dimension of the sum of thee finite-dimension subspaces that was given in the problem statement:
$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3)$.

However, that formula is not true for all subspaces. Consider this counterexample of subspaces in $\mathbb{R}^2$: $\operatorname{span}((0,1))$, $\operatorname{span}((1,0))$, $\operatorname{span}((1,1))$. Observe

$$\begin{aligned}
&\operatorname{span}((0,1)) + \operatorname{span}((1,0)) + \operatorname{span}((1,1)) \\
&= \{ v + u + w : v \in \operatorname{span}((0,1)), u \in \operatorname{span}((1,0)), w \in \operatorname{span}((1,1)) \} \\
&= \{ a_1(0,1) + a_2(1,0) + a_3(1,1) : a_1, a_2, a_3 \in \mathbf{F} \} = \operatorname{span}((0,1), (1,0), (1,1))
\end{aligned}$$

Since $(1,1) \in \operatorname{span}((0,1),(1,0))$, it can be deleted from $(0,1),(1,0),(1,1)$ without affecting the span. Therefore, $\operatorname{span}((0,1)) + \operatorname{span}((1,0)) + \operatorname{span}((1,1)) = \operatorname{span}((0,1),(1,0),(1,1)) = \operatorname{span}((0,1),(1,0)) = \mathbb{R}^2$.

However, observe the following:
$\dim(\operatorname{span}((0,1)) + \operatorname{span}((1,0)) + \operatorname{span}((1,1))) = \dim(\mathbb{R}^2) = 2 \neq 3 = 1+1+1-0-0-0+0 =$ ~~span~~ $\dim(\operatorname{span}((0,1))) + \dim(\operatorname{span}((1,0))) + \dim(\operatorname{span}((1,1))) - \dim(\{0\}) - \dim(\{0\}) - \dim(\{0\}) + \dim(\{0\}) = \dim(\operatorname{span}((0,1))) + \dim(\operatorname{span}((1,0))) + \dim(\operatorname{span}((1,1))) - \dim(\operatorname{span}((0,1)) \cap \operatorname{span}((1,0))) - \dim(\operatorname{span}((0,1)) \cap \operatorname{span}((1,1))) - \dim(\operatorname{span}((1,0)) \cap \operatorname{span}((1,1))) - \dim(\operatorname{span}((0,1)) \cap \operatorname{span}((1,0)) \cap \operatorname{span}((1,1)))$.

**20. Prove that if $V_1, V_2, V_3$ are subspaces of a finite-dimensional vector space, then $\dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2 + \dim V_3 - (\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3))/3 - (\dim((V_1+V_2) \cap V_3) + \dim((V_1+V_3) \cap V_2) + \dim((V_2+V_3) \cap V_1))/3$.**

By Theorem 2.43, $\dim((V_1+V_2)+V_3) = \dim(V_1+V_2) + \dim V_3 - \dim((V_1+V_2) \cap V_3)$. Apply the theorem again to that first term to get $\dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim((V_1+V_2) \cap V_3)$.

Similarly, $\dim((V_1+V_3)+V_2) = \dim(V_1+V_3) + \dim V_2 - \dim((V_1+V_3) \cap V_2) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_3) - \dim((V_1+V_3) \cap V_2)$.

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Once more: $\dim((V_2+V_3)+V_1) = \dim(V_2+V_3) + \dim V_1 - \dim((V_2+V_3)\cap V_1) =$
$\dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2\cap V_3) - \dim((V_2+V_3)\cap V_1)$.

Put these all together into one equation to obtain:

$$\begin{aligned}
3\dim(V_1+V_2+V_3) &= \dim((V_1+V_2)+V_3) + \dim((V_1+V_3)+V_2) + \dim((V_2+V_3)+V_1) \\
&= (\dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1\cap V_2) - \dim((V_1+V_2)\cap V_3)) \\
&\quad + (\dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1\cap V_3) - \dim((V_1+V_3)\cap V_2)) \\
&\quad + (\dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2\cap V_3) - \dim((V_2+V_3)\cap V_1)) \\
&= 3\dim V_1 + 3\dim V_2 + 3\dim V_3 - \dim(V_1\cap V_2) - \dim(V_1\cap V_3) - \dim(V_2\cap V_3) \\
&\quad - \dim((V_1+V_2)\cap V_3) - \dim((V_1+V_3)\cap V_2) - \dim((V_2+V_3)\cap V_1)
\end{aligned}$$

Divide both sides by 3 to obtain the desired result.

# 3A Vector Space of Linear Maps

## Example 3.3

**Prove that $0 \in \mathcal{L}(V,W)$ is a linear map**

Let $u, w \in V$. Then $0(u+w) = 0 = 0 + 0 = 0(u) + 0(w)$.
Let $\lambda \in \mathbf{F}$ and $v \in V$. Then $0(\lambda v) = 0 = \lambda \cdot 0 = \lambda 0(v)$.

**Prove that ~~$0 \in \mathcal{L}(V,W)$ is a linear map~~ $I \in \mathcal{L}(V)$ is a linear map**

Let $u, w \in V$. Then $I(u+w) = u+w = I(u) + I(w)$.
Let $\lambda \in \mathbf{F}$ and $v \in V$. Then $I(\lambda v) = \lambda v = \lambda I(v)$.

**Define a linear map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $(Tp)(x) = x^2 p(x)$ for every $x \in \mathbf{R}$. Prove that $T$ is a linear map.**

Let $x$ be an arbitrary real number.
Let $u, w \in \mathcal{P}(\mathbf{R})$. Then $(T(u+w))(x) = x^2(u+w)(x) = x^2u(x) + x^2w(x) =$ $(Tu)(x) + (Tw)(x) = (Tu+Tw)(x)$. Let $\lambda \in \mathbf{F}$ and $p \in \mathcal{P}(\mathbf{R})$. Then $(T\lambda p)(x) = x^2(\lambda p)(x) = \lambda x^2 p(x) = \lambda(Tp)(x) = (\lambda Tp)(x)$.

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**Define a linear map $T \in \mathcal{L}(\mathbf{F}^\infty)$ by $T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$. Prove that $T$ is a linear map.**

Let $(x_1, x_2, x_3, \ldots), (y_1, y_2, y_3, \ldots) \in \mathbf{F}^\infty$. Then $T((x_1, x_2, x_3, \ldots) + (y_1, y_2, y_3, \ldots)) = T((x_1+y_1, x_2+y_2, x_3+y_3, \ldots)) = (x_2+y_2, x_3+y_3, \ldots) = (x_2, x_3, \ldots) + (y_2, y_3, \ldots) = T((x_1, x_2, x_3, \ldots)) + T((y_1, y_2, y_3, \ldots))$.

Let $\lambda \in \mathbf{F}$ and $(x_1, x_2, x_3, \ldots) \in \mathbf{F}^\infty$. Then $T(\lambda(x_1, x_2, x_3, \ldots)) = T((\lambda x_1, \lambda x_2, \lambda x_3, \ldots)) = (\lambda x_2, \lambda x_3, \ldots) = \lambda(x_2, x_3, \ldots) = \lambda T((x_1, x_2, x_3, \ldots))$.

**Define a linear map $T \in \mathcal{L}(\mathbf{R}^3, \mathbf{R}^2)$ by $T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z)$. Prove that $T$ is a linear map.**

Let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbf{R}^3$. Then $T(x+y) = T((x_1, x_2, x_3) + (y_1, y_2, y_3)) = T(x_1+y_1, x_2+y_2, x_3+y_3) = (2(x_1+y_1) - (x_2+y_2) + 3(x_3+y_3), 7(x_1+y_1) + 5(x_2+y_2) - 6(x_3+y_3)) = ((2x_1 - x_2 + 3x_3) + (2y_1 - y_2 + 3y_3), (7x_1 + 5x_2 - 6x_3) + (7y_1 + 5y_2 - 6y_3)) = (2x_1 - x_2 + 3x_3, 7x_1 + 5x_2 - 6x_3) + (7y_1 - y_2 + 3y_3, 7y_1 + 5y_2 - 6y_3) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3) = Tx + Ty$.

Let $\lambda \in \mathbf{F}$ and $z = (z_1, z_2, z_3) \in \mathbf{R}^3$. Then $T\lambda z = T(\lambda(z_1, z_2, z_3)) = T(\lambda z_1, \lambda z_2, \lambda z_3) = (2\lambda z_1 - \lambda z_2 + 3\lambda z_3, 7\lambda z_1 + 5\lambda z_2 - 6\lambda z_3) = \lambda(2z_1 - z_2 + 3z_3, 7z_1 + 5z_2 - 6z_3) = \lambda T(z_1, z_2, z_3) = \lambda Tz$.

**Fix a polynomial $q \in \mathcal{P}(\mathbf{R})$. Define a linear map $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ by $(Tp)(x) = p(q(x))$. Prove that $T$ is a linear map.**

Let $f, g \in \mathcal{P}(\mathbf{R})$. Then $(T(f+g))(x) = (f+g)(q(x)) = f(q(x)) + g(q(x)) = (Tf)(x) + (Tg)(x) = (Tf + Tg)(x)$. Let $a \in \mathbf{R}$. Then $(Taf)(x) = (af)(q(x)) = af(q(x)) = a(Tf)(x) = (aTf)(x)$.

**1. Suppose $b, c \in \mathbf{R}$. Define $T: \mathbf{R}^3 \to \mathbf{R}^2$ by $T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$. Show that $T$ is linear if and only if $b = c = 0$.**

Assume that $b = c = 0$. Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbf{R}^3$. Then $T((x_1, x_2, x_3) + (y_1, y_2, y_3)) = T(x_1+y_1, x_2+y_2, x_3+y_3) = (2(x_1+y_1) - 4(x_2+y_2) + 3(x_3+y_3) + b, 6(x_1+y_1) + c(x_1+y_1)(x_2+y_2)(x_3+y_3)) = (2x_1 + 2y_1 - 4x_2 - 4y_2 + 3x_3 + 3y_3 + 0, 6x_1 + 6y_1 + 0) = ((2x_1 - 4x_2 + 3x_3 + 0) + (2y_1 - 4y_2 + 3y_3 + 0), (6x_1 + 0x_1x_2x_3) + (6y_1 + 0y_1y_2y_3)) =$

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$(2x_1 - 4x_2 + 3x_3 + b,\ 6x_1 + cx_1x_2x_3) + (2y_1 - 4y_2 + 3y_3 + 6,\ 6y_1 + cy_1y_2y_3) =$
$T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$.

Let $\lambda \in \mathbf{R}$ and $(v_1, v_2, v_3) \in \mathbf{R}^3$. Then $T\lambda(v_1, v_2, v_3) = T(\lambda v_1, \lambda v_2, \lambda v_3) =$
$(2\lambda v_1 - 4\lambda v_2 + 3\lambda v_3 + b,\ 6\lambda v_1 + c\lambda v_1 \lambda v_2 \lambda v_3) =$
$(\lambda(2v_1 - 4v_2 + 3v_3) + 0,\ 6\lambda v_1 + 0) = (\lambda(2v_1 - 4v_2 + 3v_3 + 0),\ 6\lambda v_1 + 0\lambda v_1 v_2 v_3) =$
$\lambda(2v_1 - 4v_2 + 3v_3 + b,\ 6v_1 + cv_1v_2v_3) = \lambda T(v_1, v_2, v_3)$. Therefore, $T$ is linear.

~~Now the converse is "If T is linear, then b = c = 0." But instead of proving this directly, we'll prove the contrapositive.~~

Now we prove the converse. Assume that $T$ is linear. Observe
$T2(1,1,1) = T(2,2,2) = (2\cdot 2 - 4\cdot 2 + 3\cdot 2 + b,\ 6\cdot 2 + 2^3 c) = (4 - 8 + 6 + b,\ 12 + 8c) =$
$(2 + b,\ 12 + 8c)$. Also notice that $2T(1,1,1) = 2(2 - 4 + 3 + b,\ 6 + c) =$
$2(1 + b,\ 6 + c) = (2 + 2b,\ 12 + 2c)$.

Then $(2 + b,\ 12 + 8c) = T2(1,1,1) = 2T(1,1,1) = (2 + 2b,\ 12 + 2c)$. So $2 + b = 2 + 2b$.
Add $-2 - b$ to both sides to get $0 = b$. Also, $12 + 8c = 12 + 2c$. Add $-12 - 2c$ to both sides to obtain $6c = 0$. Divide both sides by 6 to get $c = 0$.
Therefore, $b = c = 0$.

3. Suppose that $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exists scalars $A_{j,k} \in \mathbf{F}$ for every $j \in \{1, \dots, m\}$ and $k \in \{1, \dots, n\}$ such that
$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$ for every $(x_1, \dots, x_n)$ in $\mathbf{F}^n$.

Let $v_k$ be a vector in $\mathbf{F}^n$ such that the $k^{th}$ coordinate is 1 and all the other coordinates are 0. In other words $v_k$ is the $k^{th}$ vector in the standard basis of $\mathbf{F}^n$. Let $(A_{1,k}, \dots, A_{m,k}) = Tv_k$. Then observe the following:

$$\begin{aligned}
T(x_1, \dots, x_n) &= T(x_1, 0, \dots, 0) + \cdots + T(0, \dots, 0, x_n) = T(x_1(1, 0, \dots, 0)) + \cdots + T(x_n(0, \dots, 0, 1)) \\
&= x_1 T v_1 + \cdots + x_n T v_n = x_1(A_{1,1}, \dots, A_{m,1}) + \cdots + x_n(A_{1,n}, \dots, A_{m,n}) \\
&= (A_{1,1}x_1, \dots, A_{m,1}x_1) + \cdots + (A_{1,n}x_n, \dots, A_{m,n}x_n) \\
&= (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \cdots + A_{m,n}x_n).
\end{aligned}$$

4. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_m$ is a list of vectors in $V$ such that $Tv_1, \dots, Tv_m$ is a linearly independent list in $W$. Prove that $v_1, \dots, v_m$ is linearly independent.
Let $a_1, \dots, a_m$ be scalars such that $0 = a_1v_1 + \cdots + a_mv_m$.

Then by Theorem 3.10, $0 = T(a_1v_1 + \cdots + a_mv_m) = Ta_1v_1 + \cdots + Ta_mv_m = a_1Tv_1 + \cdots + a_mTv_m$. Since $Tv_1, \ldots, Tv_m$ is linearly independent, $a_1 = \cdots = a_m = 0$.

**5. Prove that $\mathcal{L}(V,W)$ is a vector space, as was asserted in 3.6.**

Let $R, S, T \in \mathcal{L}(V,W)$. Let $v$ be an arbitrary element of $V$. Then $(S+T)(v) = Sv + Tv = Tv + Sv = (T+S)(v)$. So $\mathcal{L}(V,W)$ is commutative. ~~By Theorem 3.8 (proved in Exercise 3A.6),~~ Also, $((R+S)+T)(v) = (R+S)v + Tv = Rv + Sv + Tv = Rv + (S+T)v = (R+(S+T))v$. Therefore $\mathcal{L}(V,W)$ is associative. Next, $(T+0)v = Tv + 0v = Tv + 0 = Tv$. Then $0$ is the additive identity of $\mathcal{L}(V,W)$.

~~Consider that~~ Define $Q: V \to W$ as $Qv = -Tv$. Then $(T+Q)v = Tv + Qv = Tv - Tv = 0$. So $\mathcal{L}(V,W)$ has an additive inverse for every element. ~~Let $\lambda \in F$. Then~~ Note that $(1T)v = 1Tv = Tv$, so $1$ is the multiplicative identity.

All that remains is to show the distributive properties of a vector space. Let $\lambda \in \mathbf{F}$. Then $(\lambda(S+T))v = \lambda(S+T)v = \lambda(Sv + Tv) = \lambda Sv + \lambda Tv = (\lambda S)v + (\lambda T)v = (\lambda S + \lambda T)v$. Also, let $a, b \in \mathbf{F}$ and notice that $((a+b)T)v = (a+b)Tv = aTv + bTv = (aT)v + (bT)v = (aT + bT)v$. Therefore, ~~Let~~ $\mathcal{L}(V,W)$ is a vector space.

**6. Prove that multiplication of linear maps has the associative, identity, and distributive properties asserted in 3.8.**

~~Let~~ First we show the associative property. Let $T_1, T_2, T_3$ be linear maps such that $T_3$ maps into the domain of $T_2$ and $T_2$ maps into the domain of $T_1$. Then $((T_1T_2)T_3)v = (T_1T_2)(T_3v) = T_1(T_2(T_3v)) = T_1((T_2T_3)v) = (T_1(T_2T_3))v$.

Next, we show the identity. Let $T \in \mathcal{L}(V,W)$, $I_V$ be the identity operator on $V$ and $I_W$ be the identity operator on $W$. ~~Then $(TI_V$~~ Observe $TI_Vv = T(I_Vv) = Tv = I_W(Tv) = I_WTv$.

Finally, we show the distributive properties. Let $Q, R \in \mathcal{L}(U,V)$ and $S, T \in \mathcal{L}(V,W)$. Then $((S+T)R)v = (S+T)(Rv) = S(Rv) + T(Rv) = SRv + TRv$. Also, $T(Q+R)v = T((Q+R)v) = T(Qv + Rv) = T(Qv) + T(Rv) = TQv + TRv = (TQ + TR)v$.

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7. Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$, and $T \in \mathcal{L}(V)$ then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

If $v = 0$ then by Theorem 3.10, $Tv = 0 = \lambda 0 = \lambda v$.
If $v \neq 0$ then by Exercise 2A.4a, the list $v$ is linearly independent. So by Theorem 2.38, it's a basis of $V$. Therefore, $Tv = \lambda v$.

8. Give an example of a function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that $\varphi av = a\varphi v$ for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but $\varphi$ is not linear.

Since $v \in \mathbf{R}^2$, we know $v = (v_1, v_2)$ for some $v_1, v_2 \in \mathbf{R}$. Define $\varphi : \mathbf{R} \to \mathbf{R}$ by

$$\varphi v = \varphi(v_1, v_2) = \begin{cases} 0 \text{ if } v_1 = 0 \\ v_1 + v_2 \text{ if } v_1 \neq 0 \end{cases}$$

If $v_1 = 0$ then $\varphi a v = \varphi a(0, v_2) = \varphi(0, av_2) = 0 = a0 = a\varphi(0, v_2) = a\varphi v$.

If $v_1 \neq 0$ then we consider two cases
If $a = 0$ then $\varphi a v = \varphi 0 = 0 = 0\varphi v = a\varphi v$
If $a \neq 0$ then $\varphi a v = \varphi a(v_1, v_2) = \varphi(av_1, av_2) = \cancel{av_1, av_2}\ av_1 + av_2 = a(v_1 + v_2) = a\varphi v$.

However, $\varphi((0,1) + (1,0)) = \varphi(1,1) = 1 + 1 = 2$ which is not equal to $\varphi(0,1) + \varphi(1,0) = 0 + (1 + 0) = 1$. Therefore, $\varphi$ is not linear.

9. Give an example of a function $\varphi : \mathbf{C} \to \mathbf{C}$ such that $\varphi(w+z) = \varphi w + \varphi z$ for all $w, z \in \mathbf{C}$ but $\varphi$ is not linear.

Recall that for every $x \in \mathbf{C}$, there's some $a, b \in \mathbf{R}$ such that $x = a + bi$.

Define $\varphi : \mathbf{C} \to \mathbf{C}$ by $\varphi(w) = \varphi(a + bi) = a$. Then $\varphi((a+bi) + (x+yi)) = \varphi((a+x) + (b+y)i) = a + x = \varphi(a+bi) + \varphi(x+yi)$. However, $\varphi(i \cdot 1) = \varphi(1i) = 0 \neq 1i = i \cdot 1 = i\varphi(1)$.

10. Prove or give a counterexample: If $q \in \mathcal{P}(\mathbf{R})$ and $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by $Tp = q \circ p$, then $T$ is a linear map.

Define $q : \mathbf{R} \to \mathbf{R}$ by $q(x) = 1$. ~~Let I be the identity~~ Let $f, h \in \mathcal{P}(\mathbf{R})$. Then $(T(f+h))(x) = (q \circ (f+h))(x) = q((f+h)(x)) = 1$. However, $(Tf + Th)(x) = (Tf)(x) + (Th)(x)$

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$= (q \circ f)(x) + (q \circ h)(x) = q(f(x)) + q(h(x)) = 1 + 1 = 2$. Since $1 \neq 2$, $T$ is not linear.

11. Suppose $V$ is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that $T$ is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

Let $v$ be an arbitrary vector in $V$. Let $S$ be an arbitrary element of $\mathcal{L}(V)$.

Assume that $T$ is a scalar multiple of the identity. Then for some $\lambda \in \mathbf{F}$, $Tv = (\lambda I)v$. Observe $(ST)v = (S(\lambda I))v = S((\lambda I)v) = S(\lambda I v) = \lambda S(Iv) = (\lambda(SI))v = (\lambda(IS))v = \lambda(IS)v = \lambda I(Sv) = ((\lambda I)S)v = (TS)v$.

Now for the converse assume that $ST = TS$. Let $v_1, \ldots, v_n$ be a basis of $V$.

For every $k \in \{1, \ldots, n\}$, consider the following. Let $f_k : V \to V$ be defined by $f_k(v) = f_k(a_1 v_1 + \cdots + a_n v_n) = a_k v_k$. Then for some $a_{1,k}, \ldots, a_{n,k} \in \mathbf{F}$, we know $Tv_k = a_{1,k}v_1 + \cdots + a_{n,k}v_n$. So $(f_k T)v_k = f_k(Tv_k) = f_k(a_{1,k}v_1 + \cdots + a_{n,k}v_n) = a_{k,k}v_k$. It follows that $a_{k,k}v_k = (f_k T)v_k = (Tf_k)v_k = T(f_k(v_k)) = Tv_k$. Therefore, $T(v_1 + \cdots + v_n) = Tv_1 + \cdots + Tv_n = a_{1,1}v_1 + \cdots + a_{n,n}v_n$.

~~Now we want to show that $a_{1,1} = \cdots = a_{n,n}$. We know $v = b_1 v_1 + \cdots b_n v_n$ for some $b_1, \ldots, b_n \in \mathbf{F}$. Let $h : V \to V$ be defined by $h(v) = h(b_1 v_1 + \cdots + b_n v_n) = b_2 v_1 + \cdots + b_n v_{n-1}$. Then $(hT)(v_1 + \cdots + v_n) = h(Tv_1 + \cdots + Tv_n) = h(a_{1,1}v_1 + \cdots + a_{n,n}v_n) = a_{2,2}v$~~

Now we want to show that $a_{1,1} = \cdots = a_{n,n}$. Since $v_1, \ldots, v_n$ is a basis of $V$, $v = b_1 v_1 + \cdots + b_n v_n$ for some $b_1, \ldots, b_n \in \mathbf{F}$. Let $h : V \to V$ be defined by $h(v) = h(b_1 v_1 + \cdots + b_n v_n) = b_2 v_1 + \cdots + b_n v_{n-1}$. Then $(hT)(v_1 + \cdots + v_n) = h(Tv_1 + \cdots + Tv_n) = h(a_{1,1}v_1 + \cdots + a_{n,n}v_n) = a_{2,2}v_1 + \cdots + a_{n,n}v_{n-1}$. Therefore, $a_{2,2}v_1 + \cdots + a_{n,n}v_{n-1} = (hT)(v_1 + \cdots + v_n) = (Th)(v_1 + \cdots + v_n) = T(h(v_1 + \cdots + v_n)) = T(v_1 + \cdots + v_{n-1}) = Tv_1 + \cdots + Tv_{n-1} = a_{1,1}v_1 + \cdots + a_{n-1,n-1}v_{n-1}$. Subtract the right side from the left to obtain

$$0 = a_{2,2}v_1 + \cdots + a_{n,n}v_{n-1} - a_{1,1}v_1 - \cdots - a_{n-1,n-1}v_{n-1}$$
$$= (a_{2,2} - a_{1,1})v_1 + \cdots + (a_{n,n} - a_{n-1,n-1})v_{n-1}.$$

Since $v_1, \ldots, v_{n-1}$ is linearly independent, all those coefficients are zero. Therefore, $a_{1,1} = \cdots = a_{n,n}$. So $Tv = T(b_1 v_1 + \cdots + b_n v_n) = b_1 Tv_1 + \cdots + b_n Tv_n = b_1 a_{1,1} v_1 + \cdots + b_n a_{n,n} v_n = a_{1,1} b_1 v_1 + \cdots + a_{1,1} b_n v_n = a_{1,1}(b_1 v_1 + \cdots + b_n v_n) = a_{1,1} v = (a_{1,1} I)v$. Therefore, $T$ is a scalar multiple of the identity.

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The proof of "If $ST=TS$, then $T$ is a scalar multiple of the identity" got quite complicated, so yesterday I looked up a solution online which I will reproduce from memory here today.

Assume that $ST=TS$. Let $v_1, \ldots, v_n$ be a basis of $V$. By Theorem 3.4, there's some linear map $R: V \to \mathbf{F}$ such that $Rv_1 = 1$. For every $w \in V$, let $S_w: V \to V$ be defined by $S_w v = (Rv)w$.

We show that $S_v$ is linear. Let $x, y \in V$ and $\lambda \in \mathbf{F}$. Then $S_v(x+y) = (R(x+y))v = (Rx+Ry)v = (Rx)v + (Ry)v = S_v x + S_v y$. Also, $S_v \lambda x = (R\lambda x)v = \lambda(Rx)v = \lambda S_v x$.

Observe $Tv = T1v = T(Rv_1)v = T(S_v v_1) = (TS_v)v_1 = (S_v T)v_1 = S_v(Tv_1) = R(Tv_1)v = R(Tv_1)Iv = (R(Tv_1)I)v$. Since $R(Tv_1) \in \mathbf{F}$, we know that $T$ is a scalar multiple of the identity.

12. Suppose $U$ is a subspace of $V$ with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$. Define $T: V \to W$ by $Tv = \begin{cases} Sv & \text{if } v \in U \\ 0 & \text{if } v \in V \text{ and } v \notin U \end{cases}$

Prove that $T$ is not a linear map on $V$.

We show a counterexample where $V = \mathbf{R}^2$, $W = \mathbf{R}$, $U = \{(0,x) : x \in \mathbf{R}\}$, and $S: U \to \mathbf{R}$ is defined by $S(0,x) = x$. Observe:
$T((1,0)+(0,1)) = T(1,1) = 0$ but $T(1,0) + T(0,1) = 0 + S(0,1) = 1$.

However, I just realized the intent of the problem was to prove that $T$ is **never** a linear map on $V$, not to simply prove that it isn't always a linear map. So we'll prove that next.

Because $S \neq 0$, there's some $u \in U$ such that $Su \neq 0$. If $u$ were $0$, then Theorem 3.10 would tell us that $Su = 0$. Therefore, $u \neq 0$. Since $u \in U$ and $u \neq 0$, we know $U \neq \{0\}$. Then $\dim U \geq 1$. By Theorem 2.37, $1 \leq \dim U \leq \dim V$. Since $U \neq V$, this becomes $1 \leq \dim U < \dim V$.

Let $m = \dim U$ and $n = \dim V$. Then $1 \leq m < n$. By Exercise 2A.4a, $u$ is linearly independent. By Theorem 2.32, $u$ can be extended to a basis of $U$, $u, u_2, \ldots, u_m$. That basis of $U$ can be extended to a basis of $V$ $u, u_2, \ldots, u_m, v_1, \ldots, v_{n-m}$.

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~~If $u+v_1 \in U$, then $v_1 = (u+v_1) - u \in U$. But since we~~

If $v_1 \in U$, then since $u_1, u_2, \ldots, u_m, v_1$ is a basis of $U$, $\dim U = m+1$. But since we know $\dim U = m$, $v_1 \notin U$.

If $u+v_1 \in U$, then $v_1 = (u+v_1) - u \in U$. Therefore, $u+v_1 \notin U$.

~~Let $x = Su$. Recall that $Su \neq 0$ so $x \notin 0$. Finally, observe the following: $T(u+v_1) = 0$ but $Tu + Tv_1 =$~~

Finally, $T(u+v_1) = 0$ but $Tu + Tv_1 = Su + 0 = Su \neq 0$. Therefore, $T$ is not a linear map on $V$.

**13.** Suppose $V$ is finite-dimensional. Prove that every linear map on a subspace of $V$ can be extended to a linear map on $V$. In other words, show that if $U$ is a subspace of $V$ and $S \in \mathcal{L}(U,W)$, then there exists $T \in \mathcal{L}(V,W)$ such that $Tu = Su$ for all $u \in U$.

Let $m = \dim U$ and $n = \dim V$. Let $u_1, \ldots, u_m$ be a basis of $U$. Then by Theorem 2.32 it can be extended to a basis of $V$, ~~$u_1, \ldots, u_m, v_{m+1}, \ldots, v_n$. So for every $v \in V$ there's some~~ $u_1, \ldots, u_m, v_{m+1}, \ldots, v_n$. So for every $v \in V$, there's some $a_1, \ldots, a_n \in \mathbf{F}$ such that $v = a_1u_1 + \cdots + a_mu_m + a_{m+1}v_{m+1} + \cdots + a_nv_n$. Let $T: V \to W$ be defined by $Tv = T(a_1u_1 + \cdots + a_mu_m + a_{m+1}v_{m+1} + \cdots + a_nv_n) = S(a_1u_1 + \cdots + a_mu_m)$.

We show that $T$ is linear. Let $u, w \in V$. Then for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{F}$, $u = \sum_{i=1}^{m} x_iu_i + \sum_{i=m+1}^{n} x_iv_i$ and $w = \sum_{i=1}^{m} y_iu_i + \sum_{i=m+1}^{n} y_iv_i$. Observe the following:

$$T(u+w) = T\left(\sum_{i=1}^{m}(x_iu_i + y_iu_i) + \sum_{i=m+1}^{n}(x_iv_i + y_iv_i)\right) = T\left(\sum_{i=1}^{m}(x_i+y_i)u_i + \sum_{i=m+1}^{n}(x_i+y_i)v_i\right)$$
$$= S\left(\sum_{i=1}^{m}(x_i+y_i)u_i\right) = S\left(\sum_{i=1}^{m}x_iu_i + \sum_{i=1}^{m}y_iu_i\right) = S\left(\sum_{i=1}^{m}x_iu_i\right) + S\left(\sum_{i=1}^{m}y_iu_i\right)$$
$$= T\left(\sum_{i=1}^{m}x_iu_i + \sum_{i=m+1}^{n}x_iv_i\right) + T\left(\sum_{i=1}^{m}y_iu_i + \sum_{i=m+1}^{n}y_iv_i\right) = Tu + Tw$$

Let $\lambda \in \mathbf{F}$. Then observe:

$$T\lambda u = T\left(\lambda\left(\sum_{i=1}^{m}x_iu_i + \sum_{i=m+1}^{n}x_iv_i\right)\right) = T\left(\sum_{i=1}^{m}\lambda x_iu_i + \sum_{i=m+1}^{n}\lambda x_iv_i\right) = S\left(\sum_{i=1}^{m}\lambda x_iu_i\right)$$
$$= S\left(\lambda\sum_{i=1}^{m}x_iu_i\right) = \lambda S\left(\sum_{i=1}^{m}x_iu_i\right) = \lambda T\left(\sum_{i=1}^{m}x_iu_i + \sum_{i=m+1}^{n}x_iv_i\right) = \lambda Tu$$

Also if $u \in U$ then $Tu = T(x_1u_1 + \cdots + x_mu_m) = S(x_1u_1 + \cdots + x_mu_m) = Su$.

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**14.** Suppose $V$ is finite-dimensional with $\dim V > 0$ and suppose $W$ is infinite-dimensional. Prove that $\mathcal{L}(V,W)$ is infinite dimensional.

Let $T_1,\dots,T_m$ be a list of linear maps in $\mathcal{L}(V,W)$. Let $v_1,\dots,v_n$ be a basis of $V$. Let $A$ be the set of every pair $(k,j)$ such that $k\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$. Then $|A|=mn$. For every $(k,j)\in A$ let $w_{k,j}=T_kv_j$. Let $L$ be a list of all such $w_{k,j}$'s.

Lemma 1: Let $T\in\mathcal{L}(V,W)$ and $v\in V$. If $T\in\operatorname{span}(T_1,\dots,T_m)$, then $Tv\in\operatorname{span}(L)$.
Since $T\in\operatorname{span}(T_1,\dots,T_m)$, for some $a_1,\dots,a_m\in\mathbf{F}$, $T=a_1T_1+\cdots+a_mT_m$. Also, for some $b_1,\dots,b_n\in\mathbf{F}$, $v=b_1v_1+\cdots+b_nv_n$. Observe:

$$Tv=(a_1T_1+\cdots+a_mT_m)v=\sum_{k=1}^{m}a_kT_kv=\sum_{k=1}^{m}\Big(a_kT_k\sum_{j=1}^{n}b_jv_j\Big)=\sum_{k=1}^{m}\Big(\sum_{j=1}^{n}a_kT_kb_jv_j\Big)=\sum_{(k,j)\in A}a_kT_kb_jv_j$$
$$=\sum_{(k,j)\in A}a_kb_jT_kv_j=\sum_{(k,j)\in A}a_kb_jw_{k,j}\in\operatorname{span}(L)$$

Because $W$ is infinite-dimensional, there must be some $w\in W$ such that $w\notin\operatorname{span}(L)$. By Theorem 3.4, there's some linear map $S\in\mathcal{L}(V,W)$ such that $Sv_1=w$. Then by the contrapositive of Lemma 1, since $Sv_1=w\notin\operatorname{span}(L)$, $S\notin\operatorname{span}(T_1,\dots,T_m)$. So there's no list of linear maps in $\mathcal{L}(V,W)$ that spans the space. Therefore, $\mathcal{L}(V,W)$ is infinite-dimensional.

**15.** Suppose $v_1,\dots,v_m$ is a linearly dependent list of vectors in $V$. Suppose also that $W\neq\{0\}$. Prove that there exist $w_1,\dots,w_m\in W$ such that no $T\in\mathcal{L}(V,W)$ satisfies $Tv_k=w_k$ for each $k=1,\dots,m$.

Because $W\neq\{0\}$, there's some $w\in W$ such that $w\neq 0$. By the linear dependence lemma (2.19), there's some $j\in\{1,\dots,m\}$ such that $v_j\in\operatorname{span}(v_1,\dots,v_{j-1})$. So for some $a_1,\dots,a_{j-1}\in\mathbf{F}$, $v_j=a_1v_1+\cdots+a_{j-1}v_{j-1}$.

For every $k\neq j$ choose $w_k=w$. For $w_j$ choose $w_j=w+(a_1+\cdots+a_{j-1})w$. We must show that there's no $T\in\mathcal{L}(V,W)$ such that $Tv_k=w_k$. Assume, for the purpose of contradicting, that there is such a $T$. Then $w+(a_1+\cdots+a_{j-1})w=w_j=Tv_j=T(a_1v_1+\cdots+a_{j-1}v_{j-1})=a_1Tv_1+\cdots+a_{j-1}Tv_{j-1}=a_1w_1+\cdots+a_{j-1}w_{j-1}=(a_1+\cdots+a_{j-1})w$. Subtract $(a_1+\cdots+a_{j-1})w$ from both sides to get $w=0$, which contradicts our earlier statement that $w\neq 0$.

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**16. Suppose $V$ is finite-dimensional with $\dim V > 1$. Prove that there exist $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.**

Let $v_1, \ldots, v_m$ be a basis of $V$. Then for every $v \in V$, $v = a_1 v_1 + \cdots + a_m v_m$ for some $a_1, \ldots, a_m \in \mathbf{F}$. Define $S: V \to V$ by $Sv = S(a_1 v_1 + \cdots + a_m v_m) = a_2 v_1 + a_1 v_2$. Define $T: V \to V$ by $Tv = T(a_1 v_1 + \cdots + a_m v_m) = a_1 v_1$.

Then $(ST)(v_1 + v_2) = S(T(1v_1 + 1v_2)) = S(1v_1) = S(1v_1 + 0v_2) = 0v_1 + 1v_2 = v_2$. Notice that $(TS)(v_1 + v_2) = T(S(1v_1 + 1v_2)) = T(1v_1 + 1v_2) = 1v_1 = v_1$. If $v_1 = v_2$, then $1v_1 - 1v_2 = v_1 - v_1 = 0$ which contradicts $v_1, \ldots, v_m$ being linearly independent. Therefore, $(ST)(v_1 + v_2) = v_2 \neq v_1 = (TS)(v_1 + v_2)$. So $ST \neq TS$.

**17. Suppose $V$ is finite-dimensional. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.**

Lemma 1: Suppose $V$ is a finite-dimensional subspace and $v_1, \ldots, v_m$ is a basis of $V$. Let $T \in \mathcal{L}(V)$. If $Tv_k = 0$ for every $k = 1, \ldots, m$, then $T = 0$. Proof: Let $v \in V$. Then $v = a_1 v_1 + \cdots + a_m v_m$ for some $a_1, \ldots, a_m \in \mathbf{F}$. So $Tv = T(a_1 v_1 + \cdots + a_m v_m) = a_1 T v_1 + \cdots + a_m T v_m = a_1 0 + \cdots + a_m 0 = 0$. Since $Tv = 0$ for every $v \in V$, $T = 0$.

First, we show that $\{0\}$ is a two-sided ideal. Let $E \in \{0\}$ (so $E = 0$) and $T \in \mathcal{L}(V)$. Fix $v$ as an arbitrary element of $V$ for the remainder of this exercise. Then $(TE)v = T(0v) = T0 = 0$ and $(ET)v = E(Tv) = 0(Tv) = 0$. So $TE = ET = 0 \in \{0\}$.

Now assume that $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$ such that $\mathcal{E} \neq \{0\}$. We show that $\mathcal{E} = \mathcal{L}(V)$. Since $\mathcal{E} \neq \{0\}$, there's some nonzero $S \in \mathcal{E}$. Let $v_1, \ldots, v_m$ be a basis of $V$. Let $k$ be an arbitrary element of $\{1, \ldots, m\}$ for the remainder of this proof. For every $i, j \in \{1, \ldots, m\}$, define $R_{i,j} \in \mathcal{L}(V)$ by $R_{i,j} v_k = \begin{cases} v_j & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$ using the linear map lemma (3.4).

By the contrapositive of Lemma 1, since $S \neq 0$ there must be some $n \in \{1, \ldots, m\}$ such that $Sv_n \neq 0$. Because $v_1, \ldots, v_m$ is a basis of $V$, there's some $a_1, \ldots, a_m \in \mathbf{F}$ such that $Sv_n = a_1 v_1 + \cdots + a_m v_m$. Since $Sv_n \neq 0$, there must be some $r \in \{1, \ldots, m\}$ such that $a_r \neq 0$.

Let $L_k = R_{r,k} S R_{k,n}$ and note that $v = b_1 v_1 + \cdots + b_m v_m$ for some $b_1, \ldots, b_m \in \mathbf{F}$. Then $L_k v = (R_{r,k} S)(R_{k,n}(b_1 v_1 + \cdots + b_m v_m)) = (R_{r,k} S)(b_1 R_{k,n} v_1 + \cdots + b_m R_{k,n} v_m) = (R_{r,k} S) b_k v_n$.

So $L_k v = (R_{r,k} S) b_k v_n = R_{r,k}(S b_k v_n) = R_{r,k}(b_k S v_n) = R_{r,k}(b_k(a_1 v_1 + \cdots + a_m v_m))$
$= R_{r,k}(b_k a_1 v_1 + \cdots + b_k a_m v_m) = b_k a_1 R_{r,k} v_1 + \cdots + b_k a_m R_{r,k} v_m = b_k a_r v_k = a_r b_k v_k$.

Then $(L_1 + \cdots + L_m)v = L_1 v + \cdots + L_m v = a_r b_1 v_1 + \cdots + a_r b_m v_m = a_r(b_1 v_1 + \cdots + b_m v_m)$
$= a_r v$. Therefore, $L_1 + \cdots + L_m = a_r I$.

Because $\mathcal{E}$ is a two-sided ideal and $S \in \mathcal{E}$, we know that $SR_{k,n} \in \mathcal{E}$. Similarly, we know that $L_k = R_{r,k} S R_{k,n} \in \mathcal{E}$. Then since subspaces are closed under addition, $a_r I = L_1 + \cdots + L_m \in \mathcal{E}$. Since subspaces are closed under scalar multiplication, $I = a_r^{-1} a_r I \in \mathcal{E}$. Now let $T \in \mathcal{L}(V)$. Then $T = IT \in \mathcal{E}$. Therefore, $\mathcal{E} = \mathcal{L}(V)$.

# 3B Null Spaces and Ranges

**1. Give an example of a linear map $T$ with $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.**

Define $T: \mathbf{R}^5 \to \mathbf{R}^2$ as $T(x_1, \ldots, x_5) = (x_1, x_2)$. Then $\operatorname{null} T = \{v \in \mathbf{R}^5 : Tv = 0\} = \{(x_1, \ldots, x_5) : x_1, \ldots, x_5 \in \mathbf{R} \text{ and } T(x_1, \ldots, x_5) = 0\} = \{(x_1, \ldots, x_5) : x_1, \ldots, x_5 \in \mathbf{R} \text{ and } (x_1, x_2) = 0\}$
$= \{(0, 0, x_3, x_4, x_5) : x_3, x_4, x_5 \in \mathbf{R}\}$. Then $(0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1)$ is a basis of $\operatorname{null} T$. Therefore, $\dim \operatorname{null} T = 3$.

Also, $\operatorname{range} T = \{Tv : v \in \mathbf{R}^5\} = \{T(x_1, \ldots, x_5) : x_1, \ldots, x_5 \in \mathbf{R}\} = \{(x_1, x_2) : x_1, x_2 \in \mathbf{R}\}$
$= \mathbf{R}^2$. So $\dim \operatorname{range} T = \dim \mathbf{R}^2 = 2$.

**2. Suppose $S, T \in \mathcal{L}(V)$ are such that $\operatorname{range} S \subseteq \operatorname{null} T$. Prove that $(ST)^2 = 0$.**

Let $v$ be an arbitrary vector in $V$. Then $(ST)^2 v = ((ST)(ST))v = (ST)((ST)v) = (ST)(S(Tv)) = S(T(S(Tv)))$. We know that $S(Tv) \in \operatorname{range} S$. Then since $\operatorname{range} S \subseteq \operatorname{null} T$, $S(Tv) \in \operatorname{null} T$. Therefore, $T(S(Tv)) = 0$. Additionally, by Theorem 3.10, we know that $S(0) = 0$. Then $(ST)^2 v = S(T(S(Tv))) = S(0) = 0$. Because $(ST)^2 v = 0$ for all $v \in V$, we know that $(ST)^2 = 0$.

# Linear Algebra Done Right 3B

3. Suppose $v_1, \dots, v_m$ is a list of vectors in $V$. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by $T(z_1, \dots, z_m) = z_1 v_1 + \cdots + z_m v_m$.

   a. What property of $T$ corresponds to $v_1, \dots, v_m$ spanning $V$?

   We show that $v_1, \dots, v_m$ spans $V$ if and only if $T$ is surjective. Observe how range $T = \{Tv : v \in \mathbf{F}^m\} = \{T(z_1, \dots, z_m) : z_1, \dots, z_m \in \mathbf{F}\} = \{z_1 v_1 + \cdots + z_m v_m : z_1, \dots, z_m \in \mathbf{F}\} = \operatorname{span}(v_1, \dots, v_m)$. We know that $v_1, \dots, v_m$ spans $V$ if and only if $\operatorname{span}(v_1, \dots, v_m) = V$. Also, $T$ is surjective if and only if range $T = V$. Since range $T = \operatorname{span}(v_1, \dots, v_m)$, these are identical conditions.

   b. What property of $T$ corresponds to $v_1, \dots, v_m$ being linearly independent?

   We show that $v_1, \dots, v_m$ is linearly independent if and only if $T$ is injective. We'll borrow from a the fact that range $T = \operatorname{span}(v_1, \dots, v_m)$. Then for every $v \in$ range $T = \operatorname{span}(v_1, \dots, v_m)$ we know that for some $a_1, \dots, a_m \in \mathbf{F}$, $v = a_1 v_1 + \cdots + a_m v_m$.

   Now assume that there's some $b_1, \dots, b_m \in \mathbf{F}$ such that there's some $k \in \{1, \dots, m\}$ where $b_k \neq 0$ and $b_1 v_1 + \cdots + b_m v_m = 0$. Then $v_1, \dots, v_m$ is not linearly independent. Also, $T(b_1, \dots, b_m) = b_1 v_1 + \cdots + b_m v_m = 0$ so null $T \neq \{0\}$. Therefore, $T$ is not injective.

   However, if there are no $b_1, \dots, b_m \in \mathbf{F}$ such that $b_1 v_1 + \cdots + b_m v_m = T(b_1, \dots, b_m) = 0$, then $v_1, \dots, v_m$ is linearly independent and null $T = \{0\}$, which implies $T$ is injective.

4. Show that $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

Let $U = \{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$. We show that $U$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$. Define $T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$ by $T(x_1, \dots, x_5) = (x_1, x_2, 0, 0)$ and define $S \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$ by $S(x_1, \dots, x_5) = (0, 0, x_3, x_4)$. Then null $T = \{(x_1, \dots, x_5) \in \mathbf{R}^5 : T(x_1, \dots, x_5) = 0\} = \{(x_1, \dots, x_5) \in \mathbf{R}^5 : (x_1, x_2, 0, 0) = 0\} = \{(0, 0, x_3, x_4, x_5) : x_3, x_4, x_5 \in \mathbf{R}\}$. So $\dim \operatorname{null} T = 3$ and $T \in U$. Similarly, null $S = \{(x_1, x_2, 0, 0, x_5) : x_1, x_2, x_5 \in \mathbf{R}\}$. So $\dim \operatorname{null} S = 3$ and $S \in U$.

Linear Algebra Done Right 3B

Consider that $(S+T)(x_1,\dots,x_5) = T(x_1,\dots,x_5) + S(x_1,\dots,x_5) = (x_1,x_2,0,0) + (0,0,x_3,x_4) = (x_1,x_2,x_3,x_4)$. Then $\operatorname{null}(S+T) = \{(x_1,\dots,x_5) \in \mathbf{R}^5 : (S+T)(x_1,\dots,x_5) = 0\} = \{(x_1,\dots,x_5) \in \mathbf{R}^5 : (x_1,x_2,x_3,x_4) = 0\} = \{(0,0,0,0,x_5) : x_5 \in \mathbf{R}\}$. So $\dim \operatorname{null}(S+T) = 1$ and $S+T \notin U$. Then $U$ is not closed under addition. Therefore, $U$ is not a subspace.

5. Give an example of $T \in \mathcal{L}(\mathbf{R}^4)$ such that range $T$ = null $T$

Define $T \in \mathcal{L}(\mathbf{R}^4)$ by $T(v_1,\dots,v_4) = (v_3,v_4,0,0)$. Then observe:

$$\begin{aligned}\operatorname{range} T &= \{T(v_1,\dots,v_4) : (v_1,\dots,v_4) \in \mathbf{R}^4\} = \{(v_3,v_4,0,0) : v_3,v_4 \in \mathbf{R}\}\\ &= \{(v_1,v_2,0,0) : v_1,v_2 \in \mathbf{R}\} = \{(v_1,v_2,v_3,v_4) \in \mathbf{R}^4 : (v_3,v_4,0,0) = 0\}\\ &= \{(v_1,\dots,v_4) \in \mathbf{R}^4 : T(v_1,\dots,v_4) = 0\} = \operatorname{null} T.\end{aligned}$$

6. Prove that there does not exist $T \in \mathcal{L}(\mathbf{R}^5)$ such that range $T$ = null $T$.

Assume, for the purpose of contradicting, that there is such a $T$. By the fundamental theorem of linear maps (3.21), ~~dim V = dim null~~

~~dim $\mathbf{R}^5$~~ $5 = \dim \mathbf{R}^5 = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim \operatorname{null} T + \dim \operatorname{null} T$
$= 2 \dim \operatorname{null} T$.

Divide both sides by 2 to get $\frac{5}{2} = \dim \operatorname{null} T$. Because $\frac{5}{2}$ is not an integer but we know that the dimension of a subspace is always an integer, we have the desired contradiction.

7. Suppose $V$ and $W$ are finite-dimensional with $2 \le \dim V \le \dim W$. Show that $\{T \in \mathcal{L}(V,W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V,W)$.

Let $U = \{T \in \mathcal{L}(V,W) : T \text{ is not injective}\}$. Let $v_1,\dots,v_m$ be a basis of $V$. Let $w_1,\dots,w_n$ be a basis of $W$. Fix $k$ as an arbitrary element of $\{1,\dots,m\}$. Define $S \in \mathcal{L}(V,W)$ by $Sv_k = \begin{cases} 0 & \text{if } k=1 \\ w_k & \text{if } k \neq 1 \end{cases}$ and $T \in \mathcal{L}(V,W)$ by $Tv_k = \begin{cases} 0 & \text{if } k=2 \\ w_k & \text{if } k \neq 2 \end{cases}$ using the linear map lemma (3.4). Observe:

$$\begin{aligned}\operatorname{null} S &= \{v \in V : Sv = 0\} = \{a_1v_1 + \cdots + a_mv_m : a_1,\dots,a_m \in \mathbf{F} \text{ and } S(a_1v_1 + \cdots + a_mv_m) = 0\}\\ &= \{a_1v_1 + \cdots + a_mv_m : a_1,\dots,a_m \in \mathbf{F} \text{ and } a_1Sv_1 + \cdots + a_mSv_m = 0\}\\ &= \{a_1v_1 + \cdots + a_mv_m : a_1,\dots,a_m \in \mathbf{F} \text{ and } a_2w_2 + \cdots + a_mw_m = 0\}\end{aligned}$$

Since $w_2,\dots,w_m$ is linearly independent, $a_2w_2 + \cdots + a_mw_m = 0$ implies $a_2 = \cdots = a_m = 0$. So $\operatorname{null} S = \{a_1v_1 : a_1 \in \mathbf{F}\} = \operatorname{span}(v_1)$. Similarly, $\operatorname{null} T = \operatorname{span}(v_2)$. Then $\dim \operatorname{null} S = \dim \operatorname{null} T = 1$ which implies that neither $S$ nor $T$ are injective. Therefore, $S, T \in U$.

However, let's take a look at null $(S+T)$:

$$\begin{aligned}
\operatorname{null}(S+T) &= \{v \in V : (S+T)v = 0\} \\
&= \{a_1v_1+\cdots+a_mv_m : a_1,\ldots,a_m \in \mathbf{F} \text{ and } (S+T)(a_1v_1+\cdots+a_mv_m)=0\} \\
&= \{a_1v_1+\cdots+a_mv_m : a_1,\ldots,a_m \in \mathbf{F} \text{ and } a_1Sv_1+\cdots+a_mSv_m+a_1Tv_1+\cdots+a_mTv_m=0\} \\
&= \{a_1v_1+\cdots+a_mv_m : a_1,\ldots,a_m \in \mathbf{F} \text{ and } 0+a_2w_2+a_3w_3+\cdots+a_mw_m+a_1w_1+0+a_3w_3+\cdots+a_mw_m=0\} \\
&= \{a_1v_1+\cdots+a_mv_m : a_1,\ldots,a_m \in \mathbf{F} \text{ and } a_1w_1+a_2w_2+2a_3w_3+\cdots+2a_mw_m=0\} \\
&= \{a_1v_1+\cdots+a_mv_m : a_1=\cdots=a_m=0\} = \{0\}.
\end{aligned}$$

So $S+T$ is injective which implies $S+T \notin U$. Then $U$ is not closed under addition. Therefore, $U$ is not a subspace of $\mathcal{L}(V,W)$.

**8. Suppose $V$ and $W$ are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $U = \{T \in \mathcal{L}(V,W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V,W)$.**

Let $m = \dim V$ and $n = \dim W$. Fix $k$ as an arbitrary element of $\{1,\ldots,m\}$. Let $v_1,\ldots,v_m$ be a basis of $V$ and $w_1,\ldots,w_n$ be a basis of $W$. Using the linear map lemma (3.4), define $S \in \mathcal{L}(V,W)$ by
$Sv_k = \begin{cases} 0 & \text{if } k=1 \text{ or } k>n \\ w_k & \text{otherwise} \end{cases}$ and $T \in \mathcal{L}(V,W)$ by $Tv_k = \begin{cases} 0 & \text{if } k=2 \text{ or } k>n \\ w_k & \text{otherwise} \end{cases}$.

~~Observe:~~
~~range $S = \{Sv : v \in V\} = \{(S+T)(a_1v_1+\cdots+a_mv_m) : a_1,\ldots,a_m \in \mathbf{F}\}$~~
~~$= \{a_1Sv_1+a_2Sv_2+a_3Sv_3+\cdots+a_mSv_m+a_1Tv_1+a_2Tv_2+a_3Tv_3+\cdots+a_mTv_m : a_1,\ldots,a_m \in \mathbf{F}\}$~~
~~$= \{$~~

Observe:

$$\begin{aligned}
\operatorname{range} S &= \{Sv : v \in V\} = \{S(a_1v_1+\cdots+a_mv_m) : a_1,\ldots,a_m \in \mathbf{F}\} \\
&= \{a_1Sv_1+a_2Sv_2+\cdots+a_mSv_m : a_1,\ldots,a_m \in \mathbf{F}\} \\
&= \{0+a_2w_2+\cdots+a_nw_n : a_2,\ldots,a_n \in \mathbf{F}\} = \operatorname{span}(w_2,\ldots,w_n).
\end{aligned}$$

Similarly, range $T = \operatorname{span}(w_1, w_3,\ldots,w_n)$. So $\dim \operatorname{range} S = \dim \operatorname{range} T = n-1$. Then neither ~~S nor T~~ range $S$ nor range $T$ is equal to $W$, which implies that neither $S$ nor $T$ is surjective. Therefore, $S, T \in U$. However:

$$\begin{aligned}
\operatorname{range}(S+T) &= \{(S+T)v : v \in V\} = \{(S+T)(a_1v_1+\cdots+a_mv_m) : a_1,\ldots,a_m \in \mathbf{F}\} \\
&= \{a_1Sv_1+a_2Sv_2+a_3Sv_3+\cdots+a_mSv_m+a_1Tv_1+a_2Tv_2+a_3Tv_3+\cdots+a_mTv_m : a_1,\ldots,a_m \in \mathbf{F}\} \\
&= \{0+a_2w_2+a_3w_3+\cdots+a_nw_n+a_1w_1+0+a_3w_3+\cdots+a_nw_n : a_1,\ldots,a_n \in \mathbf{F}\} \\
&= \{a_1w_1+a_2w_2+2a_3w_3+\cdots+2a_nw_n : a_1,\ldots,a_n \in \mathbf{F}\} = \operatorname{span}(w_1,\ldots,w_n) = W.
\end{aligned}$$

Then $S+T$ is surjective, which implies $S+T \notin U$. Then $U$ is not closed under addition. Therefore, $U$ is not a subspace of $\mathcal{L}(V,W)$.

Linear Algebra Done Right 3B

9. Suppose $T \in \mathcal{L}(V,W)$ is injective and $v_1,\dots,v_n$ is linearly independent in $V$. Prove that $Tv_1,\dots,Tv_n$ is linearly independent in $W$.

Let $a_1,\dots,a_n$ be scalars such that $0 = a_1Tv_1 + \cdots + a_nTv_n = T(a_1v_1 + \cdots + a_nv_n)$. Since $T$ is injective, $T(a_1v_1+\cdots+a_nv_n)=0$ implies that $a_1v_1+\cdots+a_nv_n=0$. Because $v_1,\dots,v_n$ is linearly independent, this implies $a_1=\cdots=a_n=0$.

10. Suppose $v_1,\dots,v_n$ spans $V$ and $T\in\mathcal{L}(V,W)$. Show that $Tv_1,\dots,Tv_n$ spans range $T$.

$$\begin{aligned}\operatorname{range} T &= \{Tv : v\in V\} = \{Tv : v \in \operatorname{span}(v_1,\dots,v_n)\} = \{T(a_1v_1+\cdots+a_nv_n) : a_1,\dots,a_n \in \mathbf{F}\}\\ &= \{a_1Tv_1+\cdots+a_nTv_n : a_1,\dots,a_n\in\mathbf{F}\} = \operatorname{span}(Tv_1,\dots,Tv_n)\end{aligned}$$

11. Suppose that $V$ is finite-dimensional and that $T\in\mathcal{L}(V,W)$. Prove that there exists a subspace $U$ of $V$ such that $U\cap \operatorname{null} T=\{0\}$ and $\operatorname{range} T = \{Tu : u\in U\}$.

Let $w_1,\dots,w_m$ be a basis of null $T$. Then we can extend it to a basis of $V$ $w_1,\dots,w_m,u_1,\dots,u_n$. Let $U=\operatorname{span}(u_1,\dots,u_n)$ which means $u_1,\dots,u_n$ is a basis of $U$.

First, we show that $U\cap\operatorname{null}T=\{0\}$. Let $v\in U\cap \operatorname{null} T$. Since $v\in\operatorname{null}T$, we know there's some $a_1,\dots,a_m\in\mathbf{F}$ such that $v=a_1w_1+\cdots+a_mw_m$. Since $v\in U$, we know there's some $b_1,\dots,b_n\in\mathbf{F}$ such that $v=b_1u_1+\cdots+b_nu_n$. Then $a_1w_1+\cdots+a_mw_m=v=b_1u_1+\cdots+b_nu_n$. Subtract $b_1u_1+\cdots+b_nu_n$ from both sides to get $a_1w_1+\cdots+a_mw_m-b_1u_1-\cdots-b_nu_n=0$. Since $w_1,\dots,w_m,u_1,\dots,u_n$ is linearly independent, this implies $a_1=\cdots=a_m=u_1=\cdots=u_n=0$. Therefore, $v=a_1w_1+\cdots+a_mw_m=0$.

Next, we show that $\operatorname{range}T=\{Tu : u\in U\}$.

$$\begin{aligned}\operatorname{range} T &= \{Tv : v\in V\} = \{T(a_1w_1+\cdots+a_mw_m+b_1u_1+\cdots+b_nu_n) : a_1,\dots,a_m,u_1,\dots,u_n\in\mathbf{F}\}\\ &= \{T(a_1w_1+\cdots+a_mw_m)+T(b_1u_1+\cdots+b_nu_n) : a_1,\dots,a_m,u_1,\dots,u_n\in\mathbf{F}\}\\ &= \{0+T(b_1u_1+\cdots+b_nu_n) : b_1,\dots,b\in\mathbf{F}\} = \{Tu : u\in U\}\end{aligned}$$

12. Suppose $T$ is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that $\operatorname{null} T = \{(x_1,x_2,x_3,x_4)\in\mathbf{F}^4 : x_1=5x_2 \text{ and } x_3=7x_4\}$. Prove that $T$ is surjective.

~~Notice that (5,1,0,0) ∈ null T. Then null T ≠ {0} so by Theorem 3.16~~ First we show that $(5,1,0,0),(0,0,7,1)$ is a basis of null $T$. Let $a_1,a_2\in\mathbf{F}$ such that $0=a_1(5,1,0,0)+a_2(0,0,7,1)=(5a_1,a_1,7a_2,a_2)$. So $a_1=a_2=0$ and the list is linearly independent.

Observe

$\operatorname{null} T = \{(x_1,x_2,x_3,x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\} = \{(5x_2,x_2,7x_4,x_4) : x_2,x_4 \in \mathbf{F}\}$

$= \{x_2(5,1,0,0) + x_4(0,0,7,1) : x_2,x_4 \in \mathbf{F}\} = \operatorname{span}((5,1,0,0),(0,0,7,1))$.

Therefore, $(5,1,0,0),(0,0,7,1)$ is a basis of $\operatorname{null} T$. So $\dim \operatorname{null} T = 2$.

By the fundamental theorem of linear maps (3.21), $4 = \dim \mathbf{F}^4 = \dim \operatorname{null} T + \dim \operatorname{range} T = 2 + \dim \operatorname{range} T$. Subtract 2 from both sides to get $\dim \operatorname{range} T = 2$. By theorem 3.18, we know $\operatorname{range} T$ is a subspace of $\mathbf{F}^2$. Then by Theorem 2.39 we know a subspace of full dimension equals the whole space so $\operatorname{range} T = \mathbf{F}^2$. Therefore, we know that $T$ is surjective.

**13. Suppose $U$ is a three-dimensional subspace of $\mathbf{R}^8$ and that $T$ is a linear map from $\mathbf{R}^8$ to $\mathbf{R}^5$ such that $\operatorname{null} T = U$. Prove that $T$ is surjective.**

By the fundamental theorem of linear maps (3.21) $8 = \dim \mathbf{R}^8 = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim U + \dim \operatorname{range} T = 3 + \dim \operatorname{range} T$. Subtract 3 from both sides to get $\dim \operatorname{range} T = 5$. Since a subspace of full dimension equals the whole space, $\operatorname{range} T = \mathbf{R}^5$. Therefore, $T$ is surjective.

**14. Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals $\{(x_1,x_2,x_3,x_4,x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.**

Assume, to the contrary, that there is such a linear map $T$ whose null space equals $U = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$. Observe

$\operatorname{null} T = U = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$

$= \{(3x_2,x_2,x_3,x_3,x_3) : x_2,x_3 \in \mathbf{F}\} = \{x_2(3,1,0,0,0) + x_3(0,0,1,1,1) : x_2,x_3 \in \mathbf{F}\}$

$= \cancel{\operatorname{span}((3,1,0,0,0),(0,0,0}\ \operatorname{span}((3,1,0,0,0),(0,0,1,1,1))$.

Therefore, $\dim \operatorname{null} T = 2$. Since $\operatorname{range} T$ is a subspace of $\mathbf{F}^2$, $\dim \operatorname{range} T \leq 2$.

By the fundamental theorem of linear maps (3.21), we know that $5 = \dim \mathbf{F}^5 = \dim \operatorname{null} T + \dim \operatorname{range} T \leq 2 + 2 = 4$, which is a contradiction.

~~**15. Suppose there exists a linear map on $V$ whose null space and range are both finite-dimensional. Prove that $V$ is finite-dimensional.**~~

~~Let $T$ be said linear map. Then there's some ~~positive~~ nonnegative integers $m,n$ such that $m = \dim \operatorname{null} T$ and $n = \dim \operatorname{range} T$. By the fundamental theorem of linear maps (3.21), $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = m + n$. Since $m+n \in \mathbf{Z}$, $V$ is finite-dimensional.~~

Linear Algebra Done Right 3B

15. Suppose there exists a linear map $T$ on $V$ whose null space and range are both finite-dimensional. Prove that $V$ is finite-dimensional.

Let $m = \dim \operatorname{null} T$ and ~~m~~ $n = \dim \operatorname{range} T$. Let $v_1, \ldots, v_m$ be a basis of null $T$. We must show that $V$ is finite-dimensional. Assume, for the purpose of contradicting, that $V$ is infinite-dimensional.

We show that there's some ~~$v_1, \ldots, v_{m+n+1}$ such~~ linearly independent list of vectors $v_1, \ldots, v_{m+n+1}$. Since $v_1, \ldots, v_m$ is a basis of null $T$, it's linearly independent. For every integer $m \leq k \leq m+n+1$, assume that $v_1, \ldots, v_k$ is linearly independent. Because $V$ is infinite dimensional, we know $v_1, \ldots, v_k$ does not span $V$. Then there must be some $v_{k+1} \in V$ such that $v_{k+1} \notin \operatorname{span}(v_1, \ldots, v_k)$. Then by the contrapositive of the linear dependence lemma (2.19), $v_1, \ldots, v_{k+1}$ is linearly independent.

~~We show that null~~ Define $S \in \mathcal{L}(\operatorname{span}(v_1, \ldots, v_{m+n+1}), V)$ using the linear map lemma (3.4) by $Sv_k = Tv_k$.

We show that null $S$ = null $T$. Let $v \in \operatorname{null} T$. Then $v = a_1 v_1 + \cdots + a_m v_m$ for some $a_1, \ldots, a_m \in \mathbf{F}$. So $0 = Tv = a_1 Tv_1 + \cdots + a_m Tv_m = a_1 Sv_1 + \cdots + a_m Sv_m = S(a_1 v_1 + \cdots + a_m v_m) = Sv$. Therefore, $v \in \operatorname{null} S$. For inclusion in the other direction let $v \in \operatorname{null} S \subseteq \operatorname{span}(v_1, \ldots, v_{m+n+1})$. Then for some $a_1, \ldots, a_{m+n+1} \in \mathbf{F}$,
$0 = Sv = S(a_1 v_1 + \cdots + a_{m+n+1} v_{m+n+1}) = a_1 Sv_1 + \cdots + a_{m+n+1} Sv_{m+n+1} = a_1 Tv_1 + \cdots + a_{m+n+1} Tv_{m+n+1} = T(a_1 v_1 + \cdots + a_{m+n+1} v_{m+n+1}) = Tv$. So $v \in \operatorname{null} T$.

We show that range $S \subseteq$ range $T$. Let $w \in \operatorname{range} S$. Then for some $v \in \operatorname{span}(v_1, \ldots, v_{m+n+1})$, $Sv = w$. So for some $a_1, \ldots, a_{m+n+1} \in \mathbf{F}$,
$w = Sv = a_1 Sv_1 + \cdots + a_{m+n+1} Sv_{m+n+1} = a_1 Tv_1 + \cdots + a_{m+n+1} Tv_{m+n+1} = Tv \in \operatorname{range} T$. Then by theorem 2.37, $\dim \operatorname{range} S \leq \dim \operatorname{range} T$.

Finally, by the fundamental theorem of linear maps (3.21),
$m+n+1 = \dim \operatorname{span}(v_1, \ldots, v_{m+n+1}) = \dim \operatorname{null} S + \dim \operatorname{range} S \leq \dim \operatorname{null} T + \dim \operatorname{range} T = m+n$. Add $-m-n$ to both sides to get $1 \leq 0$, which is our desired contradiction.

16. Suppose $V$ and $W$ are both finite-dimensional. Prove that there exists an injective linear map from $V$ to $W$ if and only if $\dim V \le \dim W$.

The contrapositive of theorem 3.22 says that if there's an injective linear map from $V$ to $W$, then $\dim V \le \dim W$.

For the converse, assume that $\dim V \le \dim W$. Let $v_1, \ldots, v_m$ be a basis of $V$ and let $w_1, \ldots, w_n$ be a basis of $W$. Since $\dim V \le \dim W$, $m \le n$. Using the linear map lemma (3.4) define $T \in \mathcal{L}(V,W)$ by $Tv_k = w_k$ for every $k \in \{1, \ldots, m\}$. Observe

$$\begin{aligned}\text{null } T &= \{v \in V : Tv = 0\} = \{a_1v_1 + \cdots + a_mv_m : a_1, \ldots, a_m \in \mathbf{F} \text{ and } T(a_1v_1 + \cdots + a_mv_m) = 0\} \\ &= \{a_1v_1 + \cdots + a_mv_m : a_1, \ldots, a_m \in \mathbf{F} \text{ and } a_1Tv_1 + \cdots + a_mTv_m = 0\} \\ &= \{a_1v_1 + \cdots + a_mv_m : a_1, \ldots, a_m \in \mathbf{F} \text{ and } a_1w_1 + \cdots + a_mw_m = 0\}\end{aligned}$$

Because $w_1, \ldots, w_m$ is linearly independent, $a_1w_1 + \cdots + a_mw_m = 0$ implies that $a_1 = \cdots = a_m = 0$. Therefore,
$\text{null } T = \{a_1v_1 + \cdots + a_mv_m : a_1 = \cdots = a_m = 0\} = \{0\}$. So $T$ is injective.

17. Suppose $V$ and $W$ are both finite-dimensional. Prove that there exists a surjective linear map from $V$ to $W$ if and only if $\dim V \ge \dim W$.

The contrapositive of theorem 3.24 says that if there's a surjective linear map from $V$ to $W$, then $\dim V \ge \dim W$.

For the converse assume that $\dim V \ge \dim W$. We know there's some basis $v_1, \ldots, v_m$ of $V$ and some basis $w_1, \ldots, w_n$ of $W$. Since $\dim V \ge \dim W$, $m \ge n$. Using the linear map lemma (3.4) define $T \in \mathcal{L}(V,W)$ by $Tv_k = \begin{cases} w_k & \text{if } k \le n \\ 0 & \text{otherwise} \end{cases}$

for every $k \in \{1, \ldots, m\}$. Observe

$$\begin{aligned}\text{range } T &= \{Tv : v \in V\} = \{T(a_1v_1 + \cdots + a_mv_m) : a_1, \ldots, a_m \in \mathbf{F}\} = \{a_1Tv_1 + \cdots + a_mTv_m : a_1, \ldots, a_m \in \mathbf{F}\} \\ &= \{a_1w_1 + \cdots + a_nw_n + 0 : a_1, \ldots, a_n \in \mathbf{F}\} = \text{span}(w_1, \ldots, w_n) = W.\end{aligned}$$

Therefore, $T$ is surjective.

18. Suppose $V$ and $W$ are finite-dimensional and that $U$ is a subspace of $V$. Prove that there exists $T \in \mathcal{L}(V,W)$ such that null $T = U$ if and only if $\dim U \ge \dim V - \dim W$.

Assume there exists such a $T$. Because range $T$ is a subspace of $W$, $\dim \text{range } T \le \dim W$. The fundamental theorem of linear maps (3.21) says $\dim V = \dim \text{null } T + \dim \text{range } T$.

Solve for dim null $T$ to obtain
$\dim U = \dim \operatorname{null} T = \dim V - \dim \operatorname{range} T \geq \dim V - \dim W$.

For the converse we start by assuming $\dim U \geq \dim V - \dim W$. There's some basis $u_1, \ldots, u_c$ of $U$. By theorem 2.32 we can extend it to a basis $u_1, \ldots, u_c, v_1, \ldots, v_m$ of $V$. Also, there's a basis $w_1, \ldots, w_n$ of $W$. So $\dim U = c$, $\dim V = c+m$ and $\dim W = n$. Then since $\dim U \geq \dim V - \dim W$, we know $c \geq c+m-n$. Flip and add $-c$ to both sides to get $m-n \leq 0$. Add $n$ to both sides to obtain $m \leq n$.

Now for every $k \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, c\}$ we may use the linear map lemma (3.4) to define $T \in \mathcal{L}(V,W)$ by $Tv_k = w_k$ and $Tu_j = 0$. Finally, observe

$$\begin{aligned}\operatorname{null} T &= \{v \in V : Tv = 0\} = \{a_1u_1 + \cdots + a_cu_c + b_1v_1 + \cdots + b_mv_m : u_1, \ldots, u_c, b_1, \ldots, b_c \in \mathbf{F} \\ &\qquad \text{and } T(a_1u_1 + \cdots + a_cu_c + b_1v_1 + \cdots + b_mv_m) = 0\} \\ &= \{a_1u_1 + \cdots + a_cu_c + b_1v_1 + \cdots + b_mv_m : a_1, \ldots, a_c, b_1, \ldots, b_m \in \mathbf{F} \text{ and} \\ &\qquad a_1Tu_1 + \cdots + a_cTu_c + b_1Tv_1 + \cdots + b_mTv_m = 0\} \\ &= \{a_1u_1 + \cdots + a_cu_c + b_1v_1 + \cdots + b_mv_m : a_1, \ldots, a_c, b_1, \ldots, b_m \in \mathbf{F} \text{ and } 0 + b_1w_1 + \cdots + b_mw_m = 0\}\end{aligned}$$

Since $w_1, \ldots, w_m$ is linearly independent, $b_1w_1 + \cdots + b_mw_m = 0$ implies $b_1 = \cdots = b_m = 0$. So $\operatorname{null} T = \{a_1u_1 + \cdots + a_cu_c + 0 : a_1, \ldots, a_c \in \mathbf{F}\} = \operatorname{span}(u_1, \ldots, u_c) = U$.

19. Suppose $W$ is finite-dimensional and $T \in \mathcal{L}(V,W)$. Prove that $T$ is injective if and only if there exists $S \in \mathcal{L}(W,V)$ such that $ST$ is the identity operator on $V$.

Assume that $T$ is injective. Then $\operatorname{null} T = \{0\}$, so $\dim \operatorname{null} T = 0$. Since range $T$ is a subspace of a finite-dimensional space, range $T$ is finite-dimensional. Then because null $T$ and range $T$ are finite-dimensional, exercise 3B.15 tells us that $V$ is finite-dimensional. Since there's an injective linear map $T$ from $V$ to $W$, the contrapositive of theorem 3.22 tells us that $\dim V \leq \dim W$.

Since $V$ is finite-dimensional, there's some basis $v_1, \ldots, v_m$ of $V$. We show that $Tv_1, \ldots, Tv_m$ is linearly independent in $W$. Let $a_1, \ldots, a_m$ be scalars such that $0 = a_1Tv_1 + \cdots + a_mTv_m = T(a_1v_1 + \cdots + a_mv_m)$. Because $T$ is injective, this implies $0 = a_1v_1 + \cdots + a_mv_m$. Since $v_1, \ldots, v_m$ is linearly independent, $a_1 = \cdots = a_m = 0$.

Then theorem 2.32 tells us that $Tv_1, \ldots, Tv_m$ can be extended to a basis $Tv_1, \ldots, Tv_m, w_1, \ldots, w_n$ of $W$. Now we can use the linear map lemma (3.4) to define $S \in \mathcal{L}(W,V)$ by $STv_k = v_k$ and $Sw_k = 0$.

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Let $v \in V$. Then there's some $b_1, \ldots, b_m \in \mathbf{F}$ such that $v = b_1v_1 + \cdots + b_mv_m$. Then $STv = S(T(b_1v_1 + \cdots + b_mv_m)) = S(b_1Tv_1 + \cdots + b_mTv_m) = b_1STv_1 + \cdots + b_mSTv_m = b_1v_1 + \cdots + b_mv_m = v$. Since $STv = v$ for every $v \in V$, $ST$ is the identity operator on $V$.

Now for the converse assume that there's some $S \in \mathcal{L}(W,V)$ such that $STv = v$ for every $v \in V$. We must show that $T$ is injective. Assume, for the purpose of contradicting, that $T$ is not injective. Then there must be some $u, v \in V$ such that $Tu = Tv$ but $u \neq v$. However, this is contradicted by $u = STu = S(Tu) = S(Tv) = STv = v$. Therefore, $T$ must be injective.

**20. Suppose $W$ is finite-dimensional and $T \in \mathcal{L}(V,W)$. Prove that $T$ is surjective if and only if there exists $S \in \mathcal{L}(W,V)$ such that $TS$ is the identity operator on $W$.**

Assume that $T$ is surjective. Then range $T = W$. Let $Tv_1, \ldots, Tv_m$ be a basis of range $T = W$. ~~Let $w \in W$. Then for some $a_1, \ldots, a_m \in \mathbf{F}$, $w = a_1Tv_1 + \cdots + a_m$~~ Using the linear map lemma (3.4), define $S \in \mathcal{L}(W,V)$ by $STv_k = v_k$. We show that $TSw = w$ for every $w \in W$. Let $w \in W$. Then for some $a_1, \ldots, a_m \in \mathbf{F}$, $w = a_1Tv_1 + \cdots + a_mTv_m$. Observe ~~$TSw = T(S(a_1$~~ $TSw = T(S(a_1Tv_1 + \cdots + a_mTv_m)) = T(a_1STv_1 + \cdots + a_mSTv_m) = T(a_1v_1 + \cdots + a_mv_m) = a_1Tv_1 + \cdots + a_mTv_m = w$. Therefore, $TS$ is the identity operator on $W$.

For the converse assume that $S \in \mathcal{L}(W,V)$ such that $TSw = w$ for every $w \in W$. We must show that $T$ is surjective. In other words, we must show that range $T = W$. Since $T \in \mathcal{L}(V,W)$, we know that range $T$ is a subspace of $W$. For inclusion in the other direction let $w \in W$. Then $w = TSw = T(Sw) \in$ range $T$. Therefore, range $T = W$ and $T$ is surjective.

**21. Suppose $V$ is finite-dimensional, $T \in \mathcal{L}(V,W)$, and $U$ is a subspace of $W$. Prove that $A = \{v \in V : Tv \in U\}$ is a subspace of $V$ and $\dim A = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$.**

Since $T0 = 0 \in U$, $0 \in A$. Let $u, v \in A$. Then $Tu \in U$ and $Tv \in U$. Since $U$ is closed under addition, $Tu + Tv = T(u+v) \in U$. Therefore, $u + v \in A$. Let $\lambda \in \mathbf{F}$. Since $U$ is closed under scalar multiplication, $\lambda Tu = T(\lambda u) \in U$. Therefore, $\lambda u \in A$. So we see that $A$ meets the conditions for a subspace (1.34).

Now we must show that $\dim A = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$. Define $S \in \mathcal{L}(A, W)$ by $Sv = Tv$ for all $v \in A$. By the fundamental theorem of linear maps (3.21), $\dim A = \dim \operatorname{null} S + \dim \operatorname{range} S$.

$\operatorname{null} S = \{v \in A : Sv = 0\} = \{v \in V : Tv \in U \text{ and } Sv = Tv = 0\}$. Since $U$ is a subspace of $W$ and $0$ is in every subspace, $Tv \in U$ is always true when $Tv = 0$. Therefore, $\operatorname{null} S = \{v \in V : Tv = 0\} = \operatorname{null} T$.

Next, we'll show that $\operatorname{range} S = U \cap \operatorname{range} T$. Let $w \in \operatorname{range} S$. Then $w = Sv$ for some $v \in A$. Since $v \in A$, $Tv \in U$. So $w = Sv = Tv \in U$. Also, $w = Sv = Tv \in \operatorname{range} T$. Therefore, $w \in U \cap \operatorname{range} T$.

For inclusion in the other direction let $w \in U \cap \operatorname{range} T$. Then since $w \in \operatorname{range} T$, $w = Tv$ for some $v \in V$. Also, $Tv = w \in U$ so $v \in A$. Because $v \in A$, we know $w = Tv = Sv \in \operatorname{range} S$. Therefore, $\operatorname{range} S = U \cap \operatorname{range} T$.

Then $\dim A = \dim \operatorname{null} S + \dim \operatorname{range} S = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$, which is our desired result.

**22.** Suppose $U$ and $V$ are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$.

Notice that $\operatorname{null} ST = \{u \in U : STu = 0\} = \{u \in U : S(Tu) = 0\} = \{u \in U : Tu \in \operatorname{null} S\}$. Because $T \in \mathcal{L}(U, V)$ and $\operatorname{null} S$ is a subspace of $V$, we can apply Exercise 2B.21 to obtain $\dim \operatorname{null} ST = \dim \{u \in U : Tu \in \operatorname{null} S\} = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T)$.

By the dimension of a sum (2A3) (solved for the intersection), we know that $\dim(\operatorname{null} S \cap \operatorname{range} T) = \dim \operatorname{null} S + \dim \operatorname{range} T - \dim(\operatorname{null} S + \operatorname{range} T)$. Since $\operatorname{range} T$ is a subspace of $\operatorname{null} S + \operatorname{range} T$, $\dim \operatorname{range} T \leq \dim(\operatorname{null} S + \operatorname{range} T)$, which implies $\dim \operatorname{range} T - \dim(\operatorname{null} S + \operatorname{range} T) \leq 0$. Could have just noticed that $\operatorname{null} S \cap \operatorname{range} T$ is a subspace of $\operatorname{null} S$ so $\dim(\operatorname{null} S \cap \operatorname{range} T) \leq \dim \operatorname{null} S$.

Therefore,

$$\begin{aligned}
\dim \operatorname{null} ST &= \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T) \\
&= \dim \operatorname{null} T + \dim \operatorname{null} S + \dim \operatorname{range} T - \dim(\operatorname{null} S + \operatorname{range} T) \\
&\leq \dim \operatorname{null} T + \dim \operatorname{null} S + 0.
\end{aligned}$$

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**23. Suppose $U$ and $V$ are finite-dimensional vector spaces and $S \in \mathcal{L}(V,W)$ and $T \in \mathcal{L}(U,V)$. Prove that $\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$.**

Let $STu_1, \ldots, STu_m$ be a basis of range $ST$. Fix $k$ as an arbitrary element of $\{1, \ldots, m\}$ for the remainder of this proof.

Since $STu_k = S(Tu_k) \in \operatorname{range} S$, we know that $\operatorname{span}(STu_1, \ldots, STu_m)$ is a subspace of range $S$. Then by theorem 2.37, $m = \dim \operatorname{span}(STu_1, \ldots, STu_m) \leq \dim \operatorname{range} S$.

Because for every $k$, $Tu_k \in \operatorname{range} T$, we know that $\operatorname{span}(Tu_1, \ldots, Tu_m)$ is a subspace of range $T$. Then by theorem 2.37 we can infer that $m = \dim \operatorname{span}(Tu_1, \ldots, Tu_k) \leq \dim \operatorname{range} T$.

Since $\dim \operatorname{range} ST = m \leq \dim \operatorname{range} S$ and $\dim \operatorname{range} ST = m \leq \dim \operatorname{range} T$, we know that $\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}$, which is our desired result.

**24a. Suppose $\dim V = 5$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \operatorname{range} TS \leq 2$.**

By the fundamental theorem of linear maps (3.21), we know that $5 = \dim V = \dim \operatorname{null} ST + \dim \operatorname{range} ST = \dim \operatorname{null}(ST + 0$, so $\dim \operatorname{null} ST = 5$. By exercise 3B.22, $5 = \dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$. Apply the fundamental theorem to $S$ and $T$ to see that $5 = \dim V = \dim \operatorname{null} S + \dim \operatorname{range} S$ and $5 = \dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. Solve for $\dim \operatorname{null} S$ and $\dim \operatorname{null} T$ to get $\dim \operatorname{null} S = 5 - \dim \operatorname{range} S$ and $\dim \operatorname{null} T = 5 - \dim \operatorname{range} T$.

Then $5 \leq \dim \operatorname{null} S + \dim \operatorname{null} T = 5 - \dim \operatorname{range} S + 5 - \dim \operatorname{range} T$. Solve for $\dim \operatorname{range} S + \dim \operatorname{range} T$ to see that
$\dim \operatorname{range} S + \dim \operatorname{range} T \leq 5$.

Assume, for the purpose of contradicting that $\min\{\dim \operatorname{range} S, \dim \operatorname{range} T\} > 2$. Then since dimension is an integer, $\min\{\dim \operatorname{range} S, \dim \operatorname{range} T\} \geq 3$. So $\dim \operatorname{range} S \geq 3$ and $\dim \operatorname{range} T \geq 3$. Then by the earlier inequality, $5 \geq \dim \operatorname{range} S + \dim \operatorname{range} T \geq 3 + 3 = 6$, which is our desired contradiction. Therefore, $\min\{\dim \operatorname{range} S, \dim \operatorname{range} T\} \leq 2$.

Linear Algebra Done Right 3B

From exercise 3B.23 we know that
dim range $TS \leq \min\{\text{dim range } T, \text{dim range } S\} \leq 2$.

24b. Give an example of $S,T \in \mathcal{L}(\mathbf{F}^5)$ with $ST=0$ and dim range $TS = 2$.

Define $S \in \mathcal{L}(\mathbf{F}^5)$ by $S(x_1,x_2,x_3,x_4,x_5) = (0,0,x_3,x_4,0)$.
Define $T \in \mathcal{L}(\mathbf{F}^5)$ by $T(x_1,x_2,x_3,x_4,x_5) = (x_3,x_4,0,0,0)$.

Then $ST(x_1,x_2,x_3,x_4,x_5) = S(x_3,x_4,0,0,0) = (0,0,0,0,0) = 0 = 0(x_1,x_2,x_3,x_4,x_5)$.

And
$$\begin{aligned}\text{range } TS &= \{TS(x_1,x_2,x_3,x_4,x_5) : (x_1,x_2,x_3,x_4,x_5) \in \mathbf{F}^5\} \\ &= \{T(0,0,x_3,x_4,0) : x_3,x_4 \in \mathbf{F}\} = \{(x_3,x_4,0,0,0) : x_3,x_4 \in \mathbf{F}\} \\ &= \{x_3(1,0,0,0,0) + x_4(0,1,0,0,0) : x_3,x_4 \in \mathbf{F}\} = \text{span}((1,0,0,0,0),(0,1,0,0,0)).\end{aligned}$$
So dim range $TS = \dim \text{span}((1,0,0,0,0),(0,1,0,0,0)) = 2$.

25. Suppose that $W$ is finite-dimensional and $S,T \in \mathcal{L}(V,W)$. Prove that null $S \subseteq$ null $T$ if and only if there exists $E \in \mathcal{L}(W)$ such that $T = ES$.

Assume that null $S \subseteq$ null $T$. ~~Let $Sa_1, \dots, Sa_m$ be a~~ Let $Tv_1, \dots, Tv_m$ be a basis of range $T$. We show that $Sv_1, \dots, Sv_m$ is linearly independent in $W$. Let $a_1, \dots, a_m$ be scalars such that $0 = a_1Sv_1 + \cdots + a_mSv_m = S(a_1v_1 + \cdots + a_mv_m)$. So $a_1v_1 + \cdots + a_mv_m \in$ null $S$. Since null $S \subseteq$ null $T$, $a_1v_1 + \cdots + a_mv_m \in$ null $T$. Then $0 = T(a_1v_1 + \cdots + a_mv_m) = a_1Tv_1 + \cdots + a_mTv_m$. Because $Tv_1, \dots, Tv_m$ is linearly independent, $a_1 = \cdots = a_m = 0$.

Now we can extend $Sv_1, \dots, Sv_m$ to a basis $Sv_1, \dots, Sv_m, Su_1, \dots, Su_n$ of range $S$. Furthermore, we can extend $Sv_1, \dots, Sv_m, Su_1, \dots, Su_n$ to a basis $Sv_1, \dots, Sv_m, Su_1, \dots, Su_n, w_1, \dots, w_j$ of $W$. With this basis we can use the linear map lemma (3.4) to define $E \in \mathcal{L}(W)$ by $ESv_k = Tv_k$, $ESu_k = 0$, and $Ew_k = 0$.

Time to show that $T = ES$. We do this by observing $ESv = Tv$ for every $v \in V$. Let $v \in V$. Since $Sv \in$ range $S$, we know that there's some scalars $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$ such that $Sv = a_1Sv_1 + \cdots + a_mSv_m + b_1Su_1 + \cdots + b_nSu_n = S(a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n)$. Then $0 = Sv - S(a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n) = S(v - a_1v_1 - \cdots - a_mv_m - b_1u_1 - \cdots - b_nu_n)$. So $v - a_1v_1 - \cdots - a_mv_m - b_1u_1 - \cdots - b_nu_n \in$ null $S$. Because null $S \subseteq$ null $T$, $v - a_1v_1 - \cdots - a_mv_m - b_1u_1 - \cdots - b_nu_n \in$ null $T$.

Then $0 = T(v - a_1v_1 - \cdots - a_mv_m - b_1u_1 - \cdots - b_nu_n) = Tv - a_1Tv_1 - \cdots - a_mTv_m - b_1Tu_1 - \cdots - b_nTu_n$.
Since $Tv \in \text{range } T$ we know there's some $c_1, \ldots, c_m \in \mathbf{F}$ such that $Tv = c_1Tv_1 + \cdots + c_mTv_m$. So observe:

$$
\begin{aligned}
0 &= Tv - a_1Tv_1 - \cdots - a_mTv_m - b_1Tu_1 - \cdots - b_nTu_n \\
&= c_1Tv_1 + \cdots + c_mTv_m - a_1Tv_1 - \cdots - a_mTv_m - b_1Tu_1 - \cdots - b_nTu_n \\
&= (c_1 - a_1)Tv_1 + \cdots + (c_m - a_m)Tv_m - b_1Tu_1 - \cdots - b_nTu_n.
\end{aligned}
$$

Since $Tv_1, \ldots, Tv_m, Tu_1, \ldots, Tu_n$ is linearly independent, all of those coefficients are zero. So $c_k = a_k$ and $b_k = 0$. Therefore, $Tv = c_1Tv_1 + \cdots + c_mTv_m = a_1Tv_1 + \cdots + a_mTv_m$.

Finally,

$$
\begin{aligned}
ESv = E(Sv) &= E(a_1Sv_1 + \cdots + a_mSv_m + b_1Su_1 + \cdots + b_nSu_n) \\
&= a_1ESv_1 + \cdots + a_mESv_m + b_1ESu_1 + \cdots + b_nESu_n \\
&= a_1Tv_1 + \cdots + a_mTv_m + 0 = Tv.
\end{aligned}
$$

Okay, we proved the biconditional in one direction. Now we must prove that if there's an $E \in \mathcal{L}(W)$ such that $T = ES$, then $\text{null } S \subseteq \text{null } T$. Let $v \in \text{null } S$. Then $Tv = ESv = E(Sv) = E(0) = 0$, where that last equality is justified by the fact that linear maps take 0 to 0 (3.10). Since $Tv = 0$, $v \in \text{null } T$.

**26. Suppose that $V$ is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\text{range } S \subseteq \text{range } T$ if and only if there exists $E \in \mathcal{L}(V)$ such that $S = TE$.**

Assume there is such an $E$. Let $Sv \in \text{range } S$. Then $Sv = TEv = T(Ev) \in \text{range } T$. Therefore, $\text{range } S \subseteq \text{range } T$.

For the converse, assume that $\text{range } S \subseteq \text{range } T$. Let $v_1, \ldots, v_m$ be a basis of $\text{null } S$. By theorem 2.32 we can extend it to a basis $v_1, \ldots, v_m, u_1, \ldots, u_n$ of $V$. For every $k \in \{1, \ldots, n\}$ notice that $Su_k \in \text{range } S$. Then since $\text{range } S \subseteq \text{range } T$, $Su_k \in \text{range } T$. So there must be some $r_k \in V$ such that $Tr_k = Su_k$. Now we can define an $E \in \mathcal{L}(V)$ by $Ev_k = 0$ and $Eu_k = r_k$ using the linear map lemma (3.4).

Let $v \in V$. Then there's some $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ such that $v = a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n$. Observe,

$$
\begin{aligned}
TEv &= TE(a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n) = a_1TEv_1 + \cdots + a_mTEv_m + b_1TEu_1 + \cdots + b_nTEu_n \\
&= 0 + b_1Tr_1 + \cdots + b_nTr_n = S(0) + b_1Su_1 + \cdots + b_nSu_n \\
&= S(a_1v_1 + \cdots + a_mv_m) + S(b_1u_1 + \cdots + b_nu_n) = S(a_1v_1 + \cdots + a_mv_m + b_1u_1 + \cdots + b_nu_n) \\
&= Sv. \text{ Therefore, } S = TE.
\end{aligned}
$$

27. Suppose $P \in \mathcal{L}(V)$, and $P^2 = P$. Prove that $V = \text{null } P \oplus \text{range } P$.

First, we show that $V \subseteq \text{null } P + \text{range } P$. Let $v \in V$. We know $Pv = P^2v$, which implies that $0 = Pv - P^2v = P(v - Pv)$. Therefore $v - Pv \in \text{null } P$. Then observe $v = (v - Pv) + Pv \in \text{null } P + \text{range } P$.

Because $V$ and null $P$ + range $P$ are mutually subspaces of each other, we know that $V = \text{null } P + \text{range } P$. Then what remains is to show that null $P$ + range $P$ is a direct sum. Let $u \in \text{null } P$ and $Pw \in \text{range } P$ such that $u + Pw = 0$. This implies $u = -Pw$. Also, we can apply $P$ to both sides of $0 = u + Pw$ to obtain $0 = P(0) = P(u + Pw) = Pu + P^2w = Pu + Pw = P(u + w)$. Therefore, $u + w \in \text{null } P$. Because $u + w$ and $u$ are both in null $P$, we know that $w = (u + w) - u \in \text{null } P$. So $Pw = 0$. Then $u = -Pw = 0$. Since $u = Pw = 0$, null $P$ + range $P$ meets the condition for a direct sum (1.45).

28. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is such that $\deg Dp = (\deg p) - 1$ for every non-constant polynomial $p \in \mathcal{P}(\mathbf{R})$. Prove that $D$ is surjective.

We show that $\mathcal{P}(\mathbf{R})$ is a subspace of range $D$. Let $p \in \mathcal{P}(\mathbf{R})$. Let $m = \deg p$. Then $p \in \mathcal{P}_m(\mathbf{R})$. We want to find a basis of $\mathcal{P}_m(\mathbf{R})$ using only vectors in range $D$. Let $z$ be the identity function on $\mathbf{R}$. For every $k \in \{1, \ldots, m+1\}$, we know that $\deg Dz^k = (\deg z^k) - 1 = k - 1$. Then $Dz^k \notin \text{span}(Dz^1, \ldots, Dz^{k-1})$ because none of the vectors in $Dz^1, \ldots, Dz^{k-1}$ have degree $k - 1 = \deg Dz^k$. So by the contrapositive of the linear dependence lemma (2.19), $Dz^1, \ldots, Dz^{m+1}$ is linearly independent in $\mathcal{P}_m(\mathbf{R})$. Then since the length of $Dz^1, \ldots, Dz^{m+1}$ is $m+1$ and $\dim \mathcal{P}_m(\mathbf{R}) = m+1$, theorem (2.38) tells us $Dz^1, \ldots, Dz^{m+1}$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

Now that we have a basis of $\mathcal{P}_m(\mathbf{R})$, we know there's some scalars $a_1, \ldots, a_{m+1}$ such that $p = a_1 Dz^1 + \cdots + a_{m+1} Dz^{m+1} = D(a_1 z^1 + \cdots + a_{m+1} z^{m+1}) \in \text{range } D$. So $\mathcal{P}(\mathbf{R})$ is a subspace of range $D$, which is mutually a subspace of $\mathcal{P}(\mathbf{R})$. Therefore, $\mathcal{P}(\mathbf{R}) = \text{range } D$ and $D$ is surjective.

30. Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Suppose $u \in V$ is not in null $\varphi$. Prove that $V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}$.

First, we show that $V \subseteq \text{null } \varphi + \{au : a \in \mathbf{F}\}$.

Let $v \in V$. If $v \in \text{null } \varphi$ then we have the desired result. Therefore, we may assume that $v \notin \text{null } \varphi$ which implies $\varphi v \neq 0$. Since $u \notin \text{null } \varphi$, $\varphi u \neq 0$. Notice $\varphi v = \frac{\varphi v \varphi u}{\varphi u} = \left(\frac{\varphi v}{\varphi u}\right)\varphi u$. Subtract $\left(\frac{\varphi v}{\varphi u}\right)\varphi u$ from both sides to get $0 = \varphi v - \left(\frac{\varphi v}{\varphi u}\right)\varphi u$.

So $0 = \varphi v - (\frac{\varphi v}{\varphi u})\varphi u = \varphi(v - (\frac{\varphi v}{\varphi u})u)$ which implies $v - (\frac{\varphi v}{\varphi u})u \in \operatorname{null}\varphi$. Finally, $v = (v - (\frac{\varphi v}{\varphi u})u) + (\frac{\varphi v}{\varphi u})u \in \operatorname{null}\varphi + \{au : a \in \mathbf{F}\}$.

Now we need only show that $\operatorname{null}\varphi + \{au : a \in \mathbf{F}\}$ is a direct sum. Let $v \in \operatorname{null}\varphi$ and $au \in \{au : a \in \mathbf{F}\}$ such that $0 = v + au$. Apply $\varphi$ to both sides to get $0 = \varphi(0) = \varphi v + a\varphi u = a\varphi u$. Since $\varphi u \neq 0$, the previous equation implies that $a = 0$. Then $0 = v + au = v$. Since $v = au = 0$, $\operatorname{null}\varphi + \{au : a \in \mathbf{F}\}$ meets the condition for a direct sum (1.45).

31. Suppose $V$ is finite-dimensional, $X$ is a subspace of $V$, and $Y$ is a finite-dimensional subspace of $W$. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$ if and only if $\dim X + \dim Y = \dim V$.

Assume there is such a $T$. Then by the fundamental theorem of linear maps (3.21), $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim X + \dim Y$.

For the converse assume that $\dim X + \dim Y = \dim V$. Let $m = \dim X$ and $n = \dim Y$ so $\dim V = m + n$. We know there's some basis $u_1, \dots, u_m$ of $X$. We can extend it to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of $V$. Also, there must be some basis $w_1, \dots, w_n$ of $Y$. Define $T \in \mathcal{L}(V, W)$ using the linear map lemma (3.4) by $Tu_k = 0$ and $Tv_k = w_k$.

$$\begin{aligned}
\operatorname{null} T = \{v \in V : Tv = 0\} &= \{a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n : a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F} \\
&\qquad \text{and } T(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) = 0\} \\
&= \{a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n : a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F} \\
&\qquad \text{and } a_1Tu_1 + \cdots + a_mTu_m + b_1Tv_1 + \cdots + b_nTv_n = 0\} \\
&= \{a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n : a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F} \\
&\qquad \text{and } 0 + b_1w_1 + \cdots + b_nw_n = 0\}
\end{aligned}$$

Since $w_1, \dots, w_n$ is linearly independent, $b_1w_1 + \cdots + b_nw_n = 0$ implies $b_1 = \cdots = b_n = 0$. So

$$\begin{aligned}
\operatorname{null} T &= \{a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n : a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F} \text{ and } b_1 = \cdots = b_n = 0\} \\
&= \{a_1u_1 + \cdots + a_mu_m + 0 : a_1, \dots, a_m \in \mathbf{F}\} = \operatorname{span}(u_1, \dots, u_m) = X.
\end{aligned}$$

$$\begin{aligned}
\operatorname{range} T = \{Tv : v \in V\} &= \{T(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) : a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}\} \\
&= \{a_1Tu_1 + \cdots + a_mTu_m + b_1Tv_1 + \cdots + b_nTv_n : a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}\} \\
&= \{0 + b_1w_1 + \cdots + b_nw_n : b_1, \dots, b_n \in \mathbf{F}\} = \operatorname{span}(w_1, \dots, w_n) = Y.
\end{aligned}$$

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32. Suppose $V$ is finite-dimensional with $\dim V > 1$. Show that if $\varphi : \mathcal{L}(V) \to \mathbf{F}$ is a linear map such that $\varphi(ST) = \varphi(S)\varphi(T)$ for all $S, T \in \mathcal{L}(V)$, then $\varphi = 0$.

First, we show that $\operatorname{null} \varphi \neq \{0\}$. Exercise 3A.16 tells us there's some $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$. Then $ST - TS \neq 0$. Since $\varphi(ST) = \varphi(S)\varphi(T) = \varphi(T)\varphi(S) = \varphi(TS)$, we can infer that $0 = \varphi(ST) - \varphi(TS) = \varphi(ST - TS)$ which implies $ST - TS \in \operatorname{null} \varphi$. Because $ST - TS \neq 0$ and $ST - TS \in \operatorname{null} \varphi$, we know that $\operatorname{null} \varphi \neq \{0\}$.

Next, notice that $\operatorname{range} \varphi$ is a subspace of $\mathbf{F}$. Then by theorem 2.37, $\dim \operatorname{range} \varphi \leq \dim \mathbf{F} = 1$. Since the dimension of a finite-dimensional vector space is a nonnegative integer, the prior inequality implies $\dim \operatorname{range} \varphi \in \{0, 1\}$.

Assume, with the goal of contradicting, that $\dim \operatorname{range} \varphi = 1$. Let $\varphi T$ be a basis of $\operatorname{range} \varphi$. Then $\varphi T \neq 0$ which implies $T \neq 0$ and $T \notin \operatorname{null} \varphi$. Therefore, $\operatorname{null} \varphi \neq \mathcal{L}(V)$. Since $\operatorname{null} \varphi \neq \{0\}$ and $\operatorname{null} \varphi \neq \mathcal{L}(V)$, Exercise 3A.17 tells us that $\operatorname{null} \varphi$ is not a two-sided ideal of $\mathcal{L}(V)$. Then there must be some $E \in \operatorname{null} \varphi$ and some $R \in \mathcal{L}(V)$ such that $ER \notin \operatorname{null} \varphi$ or $RE \notin \operatorname{null} \varphi$. Observe $\varphi(ER) = \varphi(E)\varphi(R) = 0 \cdot \varphi(R) = 0$ and $\varphi(RE) = \varphi(R)\varphi(E) = \varphi(R) \cdot 0 = 0$. Therefore, $ER \in \operatorname{null} \varphi$ and $RE \in \operatorname{null} \varphi$ which contradicts $ER \notin \operatorname{null} \varphi$ or $RE \notin \operatorname{null} \varphi$. Then $\dim \operatorname{range} \varphi \neq 1$.

So it must be the case that $\dim \operatorname{range} \varphi = 0$. Then $\operatorname{range} \varphi = \{0\}$ which implies that $\varphi = 0$.

33. Suppose that $V$ and $W$ are real vector spaces and $T \in \mathcal{L}(V, W)$. Define $T_C : V_C \to W_C$ by $T_C(u + iv) = Tu + iTv$ for all $u, v \in V$.

a. Show that $T_C$ is a (complex) linear map from $V_C$ to $W_C$.

Let $r + is, u + iv \in V_C$. Then $T_C(r + is + u + iv) = T_C((r+u) + i(s+v)) = T(r+u) + iT(s+v)$ $= Tr + Tu + iTs + iTv = Tr + iTs + Tu + iTv = T_C(r + is) + T_C(u + iv)$. Let $\lambda \in \mathbf{F}$. Then $T_C(\lambda(u + iv)) = T_C(\lambda u + i\lambda v) = T\lambda u + iT\lambda v = \lambda(Tu + iTv) = \lambda T_C(u + iv)$.

Should have used $a + bi \in \mathbf{C}$ instead of $\lambda \in \mathbf{F}$.

b. Show that $T_C$ is injective if and only if $T$ is injective.

Assume $T_C$ is injective. Let $v \in \operatorname{null} T$. Observe $T_C(v + i0) = Tv + iT(0) = Tv = 0$. Since $T_C$ is injective, this implies that $0 = v + i0 = v$. Therefore, $T$ is injective.

~~Now for the converse assume T is injective. Let $u+iv \in$ null $T_c$. Observe $0 = T_c(u+iv) = Tu + iTv$. Subtract Tu from both sides to get Subtract Tu from both sides to get $iTv = -Tu$. Square both sides to get $(Tu)^2 = (iTv)^2 = i^2 \cdot (Tv)^2 = -(Tv)^2$ which implies~~

Now for the converse assume T is injective. Let $u+iv \in$ null $T_c$. Observe $0 = T_c(u+iv) = Tu + iTv$. Then $Tu = Tv = 0$. Since T is injective, this implies $u = v = 0$. Then $u+iv = 0+i0 = 0$. Therefore, $T_c$ is injective.

c. Show that range $T_c = W_c$ if and only if range $T = W$.

Assume that range $T_c = W_c$. Let $w \in W$. Because $w \in W_c =$ range $T_c$, there must be some $T_c(u+iv) \in$ range $T_c$ such that $w = T_c(u+iv) = Tu + iTv$. Since ~~we know the im~~ $w = w + i0$, $Tv = 0$. So $w = Tu + iTv = Tu \in$ range T. Therefore, $W \subseteq$ range T. Since we also know range $T \subseteq W$, range $T = W$.

Now for the converse assume range $T = W$. Let $w + iu \in W_c$. Since ~~[illegible]~~ $w, u \in W =$ range T, there's some $r, s \in V$ such that $Tr = w$ and $Ts = u$. Then $w + iu = Tr + iTs = T_c(r+is) \in$ range $T_c$. So $W_c \subseteq$ range $T_c$. Since we also know range $T_c \subseteq W_c$, range $T_c = W_c$.

# 3C Matrices

## 1. Suppose $T \in \mathcal{L}(V,W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least dim range T nonzero entries.

Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ be arbitrary bases of V and W respectively. Let $A = \mathcal{M}(T, v_1, \ldots, v_n, w_1, \ldots, w_m)$. Note how range $T = \{Tv : v \in V\} = \{T(a_1v_1 + \cdots + a_nv_n : a_1 \ldots a_n \in F\} = \{a_1Tv_1 + \cdots + a_nTv_n : a_1, \ldots, a_n \in F\} = \operatorname{span}(Tv_1, \ldots, Tv_n)$. Let $j =$ dim range T. ~~Let b be the nu~~ Let b be the number of nonzero vectors in $Tv_1, \ldots, Tv_n$.

~~Assume~~

~~Assume, with the goal of contradicting, that $b < j$. We know we can remove all zeros from a list of vectors without changing the span. Therefore, there's some $u_1, \ldots, u_b$~~

We know we can remove all nonzero vectors from a list of vectors without affecting the span. Therefore, there's some ~~[illegible]~~ distinct

$i_1, \dots, i_b \in \{1, \dots, n\}$ such that $\operatorname{span}(Tv_{i_1}, \dots, Tv_{i_b}) = \operatorname{span}(Tv_1, \dots, Tv_n)$ and $Tv_{i_k} \neq 0$. Then $j = \dim \operatorname{range} T = \dim \operatorname{span}(Tv_1, \dots, Tv_n) = \dim \operatorname{span}(Tv_{i_1}, \dots, Tv_{i_b}) \leq b$. For every $k \in \{1, \dots, b\}$ notice how $0 \neq Tv_{i_k} = A_{1,i_k} w_1 + \dots + A_{m,i_k} w_m$. Therefore, there's some $c_k \in \{1, \dots, m\}$ such that $A_{c_k, i_k} \neq 0$. Therefore, there are ~~b = range T~~ $b \geq \dim \operatorname{range} T$ ~~entries~~ nonzero entries of $A$.

**2. Suppose $V$ and $W$ are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of $V$ and a basis of $W$ such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.**

Assume there are some bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$ of $V$ and $W$ respectively such that $\mathcal{M}(T, v_1, \dots, v_n, w_1, \dots, w_m) = A$ has all its entries equal to 1. Let $w = w_1 + \dots + w_n$. Since $w$ is a linear combination of a linearly independent list with nonzero scalars, $w \neq 0$.

We show that $\operatorname{range} T \subseteq \operatorname{span}(w)$. Let $Tv \in \operatorname{range} T$. Then for some scalars $a_1, \dots, a_n \in \mathbf{F}$, $v = a_1 v_1 + \dots + a_n v_n$. Observe:

$$Tv = T\left(\sum_{k=1}^{n} a_k v_k\right) = \sum_{k=1}^{n} a_k T v_k = \sum_{k=1}^{n} a_k \left(\sum_{j=1}^{m} A_{j,k} w_j\right) = \sum_{k=1}^{n} a_k \left(\sum_{j=1}^{m} w_j\right) = \left(\sum_{k=1}^{n} a_k\right)(w_1 + \dots + w_m)$$
$$= \left(\sum_{k=1}^{n} a_k\right) w \in \operatorname{span}(w).$$

Then $\dim \operatorname{range} T \leq \dim \operatorname{span}(w) = 1$. Because
$$0 \neq nw = n \sum_{j=1}^{m} w_j = \sum_{k=1}^{n} \left(\sum_{j=1}^{m} A_{k,j} w_j\right) = \sum_{k=1}^{n} T v_k = Tv_1 + \dots + Tv_n = T(v_1 + \dots + v_n) \in \operatorname{range} T,$$
we know that $\operatorname{range} T \neq \{0\}$. Therefore, $\dim \operatorname{range} T = 1$.

For the converse assume that $\dim \operatorname{range} T = 1$. Let $Tv \in \operatorname{range} T$ such that $Tv \neq 0$. ~~Then -Tv ≠ 0 and -Tv = T(-v) ∈ range T as well. Since -Tv ≠ 0, the list -Tv is linearly independent in W.~~ Then the list $Tv$ is linearly independent so we can extend it to a basis $Tv, w_2, \dots, w_m$ of $W$. Let $w_1 = Tv$. ~~Let u~~ For every $k \in \{1, \dots, m-1\}$ let $u_k = w_k - w_{k+1}$.

We show that $u_1, \dots, u_{m-1}, w_m$ is linearly independent. Let $a_1, \dots, a_m$ be scalars such that $0 = a_1 u_1 + \dots + a_{m-1} u_{m-1} + a_m w_m$
$$= \sum_{k=1}^{m-1} a_k u_k + a_m w_m = \sum_{k=1}^{m-1} a_k (w_k - w_{k+1}) + a_m w_m = \sum_{k=1}^{m-1} (a_k w_k - a_k w_{k+1}) + a_m w_m.$$

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So $0=\sum_{k=1}^{m-1}(a_kw_k-a_kw_{k+1})+a_mw_m=a_1w_1+\sum_{k=2}^{m}(-a_{k-1}w_k+a_kw_k)=a_1w_1+\sum_{k=2}^{m}(a_k-a_{k-1})w_k$.
Since $w_1,\dots,w_m$ is linearly independent, we know all the coefficients are 0. So $a_1=0$ and for every $k\in\{2,\dots,m\}$ $a_k-a_{k-1}=0$ which implies $a_k=a_{k-1}$. Then $0=a_1=\cdots=a_m$ by induction.

Because $u_1,\dots,u_{m-1},w_m$ is a linearly independent list of vectors in $W$ with length $m=\dim W$, theorem 2.38 tells us $u_1,\dots,u_{m-1},w_m$ is a basis of $W$. Also, observe the following:

$$\begin{aligned}u_1+\cdots+u_{m-1}+w_m &= (w_1-w_2)+(w_2-w_3)+\cdots+(w_{m-2}-w_{m-2})+(w_{m-1}-w_m)+w_m\\ &= w_1+(w_2-w_2)+(w_3-w_3)+\cdots+(w_{m-1}-w_{m-1})+(w_m-w_m)\\ &= w_1+0=w_1=Tv.\end{aligned}$$

Because $Tv\neq 0$ and linear maps take 0 to 0 (3.10), we know that $v\neq 0$. ~~Then the list v is linearly independent and can be extended to a basis v, v_2, ..., v_n of V. Let~~ The fundamental theorem of linear maps (3.21) tells us $\dim V=\dim\operatorname{null}T+\dim\operatorname{range}T=\dim\operatorname{null}T+1$. Let $n=\dim V$ and solve for $\dim\operatorname{null}T$ to get $\dim\operatorname{null}T=\dim V-1=n-1$. Now we know there's some basis $v_2,\dots,v_n$ of $\operatorname{null}T$. Because $Tv\neq 0$, $v\notin\operatorname{null}T=\operatorname{span}(v_2,\dots,v_n)$. So $v,v_2,\dots,v_n$ is linearly independent. By theorem 2.38, $v,v_2,\dots,v_n$ is a basis of $V$.

We show that $v,v_2+v,\dots,v_n+v$ is linearly independent. Let $a_1,\dots,a_n$ be scalars such that $0=a_1v+a_2(v_2+v)+\cdots+a_n(v_n+v)=a_1v+a_2v_2+a_2v+\cdots+a_nv_n+a_nv=(a_1+\cdots+a_n)v+a_2v_2+\cdots+a_nv_n$. Since $v,v_2,\dots,v_n$ is linearly independent, all those coefficients are 0. So $a_2=\cdots=a_n=0$ and $0=a_1+\cdots+a_n=a_1+0=a_1$. Therefore, $v,v_2+v,\dots,v_n+v$ is linearly independent. By theorem 2.38, its a basis of $V$.

Let $A=\mathcal{M}(T,(v,v_2+v,\dots,v_n+v),(u_1,\dots,u_{m-1},w_m))$. Then $u_1+\cdots+u_{m-1}+w_m=Tv=A_{1,1}u_1+\cdots+A_{m-1,1}u_{m-1}+A_{m,1}w_m$. Therefore, $A_{1,1}=\cdots=A_{m,1}=1$. For every $k\in\{2,\dots,n\}$ notice that $T(v_k+v)=Tv_k+Tv=0+Tv=Tv$. So $u_1+\cdots+u_{m-1}+w_m=Tv=T(v_k+v)=A_{1,k}u_1+\cdots+A_{m-1,k}u_{m-1}+A_{m,k}w_m$. Therefore, $A_{1,k}=\cdots=A_{m,k}=1$. We've show that all entries of $A$ equal 1, so we have our desired result.

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3. Suppose $v_1, \ldots, v_n$ is a basis of $V$ and $w_1, \ldots, w_m$ is a basis of $W$.

a. Show that if $S, T \in \mathcal{L}(V, W)$, then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Let $A = \mathcal{M}(S)$, $B = \mathcal{M}(T)$ and $C = \mathcal{M}(S+T)$. We must show that $C = A+B$. For every $k \in \{1, \ldots, n\}$ observe the following:

$$\sum_{j=1}^{m} C_{j,k} w_j = (S+T)v_k = Sv_k + Tv_k = \sum_{j=1}^{m} A_{j,k} w_j + \sum_{j=1}^{m} B_{j,k} w_j = \sum_{j=1}^{m} (A_{j,k} w_j + B_{j,k} w_j)$$

$$= \sum_{j=1}^{m} (A_{j,k} + B_{j,k}) w_j = \sum_{j=1}^{m} (A+B)_{j,k} w_j$$

Since $w_1, \ldots, w_m$ is linearly independent, this implies that for every $j \in \{1, \ldots, m\}$, $C_{j,k} = (A+B)_{j,k}$ which is the desired result.

b. Show that if $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$ then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

Let $A = \mathcal{M}(\lambda T)$ and $B = \mathcal{M}(T)$. We must show that $A = \lambda B$. For every $k \in \{1, \ldots, n\}$ observe:

$$\sum_{j=1}^{m} A_{j,k} w_j = (\lambda T) v_k = \lambda (T v_k) = \lambda \sum_{j=1}^{m} B_{j,k} w_j = \sum_{j=1}^{m} \lambda B_{j,k} w_j$$

Since $w_1, \ldots, w_m$ is linearly independent, this implies that for every $j \in \{1, \ldots, m\}$, $A_{j,k} = \lambda B_{j,k} = (\lambda B)_{j,k}$ which is the desired result.

5. Suppose $V$ and $W$ are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of $V$ and a basis of $W$ such that with respect to these bases, all entries of $\mathcal{M}(T)$ are $0$ except that the entries in row $k$, column $k$, equal $1$ if $1 \leq k \leq \dim \operatorname{range} T$.

Let $Tu_1, \ldots, Tu_c$ be a basis of range $T$. Then $c = \dim \operatorname{range} T$. We can extend it to a basis $Tu_1, \ldots, Tu_c, w_1, \ldots, w_m$ of $W$. ~~Since Tu_1, ..., Tu_c is linearly independent Tu_k ≠ 0 which implies~~

We show that $u_1, \ldots, u_c$ is linearly independent in $V$. Let $a_1, \ldots, a_c$ be scalars such that $0 = a_1 u_1 + \cdots + a_c u_c$. Apply $T$ to both sides to get $0 = T(0) = a_1 T u_1 + \cdots + a_c T u_c$. Since $Tu_1, \ldots, Tu_c$ is linearly independent, $a_1 = \cdots = a_c = 0$.

~~Then we can extend it to a basis u_1, ..., u_c, v_1, ..., v_n of V.~~ Let $v_1, \ldots, v_n$ be a basis of null $T$. Because $Tu_k \neq 0$, $u_k \notin \operatorname{null} T = \operatorname{span}(v_1, \ldots, v_n)$. Therefore, the list $u_1, \ldots, u_c, v_1, \ldots, v_n$ is linearly independent. This reasoning isn't quite sufficien but there's another way to infer the existence of an equivalent basis.

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The fundamental theorem of linear maps (3.21) tells us that $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = n + c = c + n$. Then since $u_1, \ldots, u_c, v_1, \ldots, v_n$ is a linearly independent list of length $c+n = \dim V$, $u_1, \ldots, u_c, v_1, \ldots, v_n$ is a basis of $V$.

Let $A = \mathcal{M}(T, (u_1, \ldots, u_c, v_1, \ldots, v_n), (Tu_1, \ldots, Tu_c, w_1, \ldots, w_m))$. Let $k$ be an arbitrary element of $\{1, \ldots, n\}$. Observe how for every $k \le c = \dim \operatorname{range} T$,

$$Tu_k = A_{1,k} Tu_1 + \cdots + A_{c,k} Tu_c + A_{c+1,k} w_1 + \cdots + A_{c+m,k} w_m$$

Since $Tu_1, \ldots, Tu_c, w_1, \ldots, w_m$ is linearly independent, this implies that $A_{k,k} = 1$ and all the other $A_{\cdot,k}$'s are zero.

For every $k > c$ observe

$0 = Tv_k = A_{1,k} Tu_1 + \cdots + A_{c,k} Tu_c + A_{c+1,k} w_1 + \cdots + A_{c+m,k} w_m$, which implies all $A_{\cdot,k}$'s are zero.

**6. Suppose $v_1, \ldots, v_m$ is a basis of $V$ and $W$ is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove there exists a basis $w_1, \ldots, w_n$ of $W$ such that all entries in the first column of $\mathcal{M}(T)$ are $0$ except for possibly a $1$ in the first row, first column.**

Case 1: $Tv_1 = 0$ ($v_1 \in \operatorname{null} T$)

Let $w_1, \ldots, w_n$ be a basis of $W$. Let $A = \mathcal{M}(T)$. Then $0 = Tv_1 = A_{1,1} w_1 + \cdots + A_{n,1} w_n$. Since $w_1, \ldots, w_n$ is linearly independent, $A_{1,1} = \cdots = A_{n,1} = 0$ which means all the entries of $A = \mathcal{M}(T)$ in the first column are $0$.

Case 2: $Tv_1 \neq 0$ ($v_1 \notin \operatorname{null} T$)

Then the list $Tv_1$ is linearly independent in $W$ and can be extended to a basis $Tv_1, w_2, \ldots, w_n$ of $W$. Let $A = \mathcal{M}(T, (v_1, \ldots, v_m), (Tv_1, w_2, \ldots, w_n))$. Then $1 Tv_1 = Tv_1 = A_{1,1} Tv_1 + A_{2,1} w_2 + \cdots + A_{n,1} w_n$. Since $Tv_1, w_2, \ldots, w_n$ is linearly independent, this implies that $A_{1,1} = 1$ and $A_{2,1} = \cdots = A_{n,1} = 0$ which is the desired result.

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**Lemma 3C.J1** if $V$ is finite-dimensional and $T \in \mathcal{L}(V,W)$, then there exists a basis $v_1, \ldots, v_n, u_1, \ldots, u_c$ of $V$ such that and $Tu_1, \ldots, Tu_c$ is a basis of range $T$. ($v_1, \ldots, v_n$ is a basis of null$(T)$)

This lemma can be used in place of my insufficient logic in Exercise 3C.5 to establish the existence of such a basis.

Let $v_1, \ldots, v_n$ be a basis of null $T$. Then we may extend it to a basis $v_1, \ldots, v_n, u_1, \ldots, u_c$ of $V$. To see that $Tu_1, \ldots, Tu_c$ is linearly independent, let $a_1, \ldots, a_c$ be scalars such that $0 = T(0) = a_1 Tu_1 + \cdots + a_c Tu_c = T(a_1u_1 + \cdots + a_cu_c)$. So $a_1u_1 + \cdots + a_cu_c \in$ null $T$. Then there's some scalars $b_1, \ldots, b_n$ such that $a_1u_1 + \cdots + a_cu_c = b_1v_1 + \cdots + b_nv_n$ which implies $b_1v_1 + \cdots + b_nv_n - a_1u_1 - \cdots - a_cu_c = 0$. Since $v_1, \ldots, v_n, u_1, \ldots, u_c$ is linearly independent, $b_1 = \cdots = b_n = a_1 = \cdots = a_c = 0$.

Because $v_1, \ldots, v_n, u_1, \ldots, u_c$ is a basis of $V$, we know $\dim V = n + c$. By the fundamental theorem of linear maps (3.21), $n + c = \dim V = \dim \text{null } T + \dim \text{range } T = n + \dim \text{range } T$. Solve for dim range $T$ to get $\dim \text{range } T = c$. Since $Tu_1, \ldots, Tu_c$ is a linearly independent list of length $c = \dim \text{range } T$, theorem 2.38 tells us that $Tu_1, \ldots, Tu_c$ is a basis of range $T$.

**7.** Suppose $w_1, \ldots, w_n$ is a basis of $W$ and $V$ is finite-dimensional. Suppose $T \in \mathcal{L}(V,W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of $V$ such that all entries in the first row of $\mathcal{M}(T)$ are $0$ except for possibly a $1$ in the first row, first column.

~~For every $v \in V$ we know there's some scalars $a_{v,1}, \ldots, a_{v,n}$ such that $Tv = a$~~

Case 1: There's some $u \in V$ such that $Tu = 1w_1 + a_2w_2 + \cdots + a_nw_n$ for some scalars $a_2, \ldots, a_n$.

If $u = 0$, then $0 = T(0) = Tu = 1w_1 + a_2w_2 + \cdots + a_nw_n$ contradicts $w_1, \ldots, w_n$ being linearly independent. Therefore, $u \neq 0$. Then the list $u$ can be extended to a basis $u, v_2, \ldots, v_m$ of $V$. Let $k$ be an arbitrary element of $\{2, \ldots, m\}$. For every $k$, we know there's some scalars $b_{1,k}, \ldots, b_{n,k}$ such that $Tv_k = b_{1,k}w_1 + \cdots + b_{n,k}w_n$.

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We show that $u, v_2 - b_{1,2}u, \ldots, v_m - b_{1,m}u$ is linearly independent. Let $c_1, \ldots, c_m$ be scalars such that

$$0 = c_1 u + c_2(v_2 - b_{1,2}u) + \cdots + c_m(v_m - b_{1,m}u) = c_1 u + \sum_{k=2}^{m} c_k(v_k - b_{1,k}u)$$

$$= c_1 u + \sum_{k=2}^{m}(c_k v_k - c_k b_{1,k}u) = c_1 u - \sum_{k=2}^{m} c_k b_{1,k} u + \sum_{k=2}^{m} c_k v_k$$

$$= \left(c_1 - \sum_{k=2}^{m} c_k b_{1,k}\right)u + \sum_{k=2}^{m} c_k v_k$$

Since $u, v_2, \ldots, v_m$ is linearly independent, all those coefficients are 0. So $c_k = 0$ and $0 = c_1 - \sum_{k=2}^{m} c_k b_{1,k} = c_1 - 0 = c_1$.

Let $A = \mathcal{M}(T, (u, v_2 - b_{1,2}u, \ldots, v_m - b_{1,m}u), (w_1, \ldots, w_n))$. Then $1w_1 + a_2w_2 + \cdots + a_nw_n = Tu = A_{1,1}w_1 + \cdots + A_{n,1}w_n$ which implies $A_{1,1} = 1$. Also,

$$A_{1,k}w_1 + \cdots + A_{n,k}w_n = T(v_k - b_{1,k}u) = Tv_k - b_{1,k}Tu = \sum_{j=1}^{n} b_{j,k}w_j - b_{1,k}\left(1w_1 + \sum_{j=2}^{n} a_j w_j\right)$$

$$= b_{1,k}w_1 - b_{1,k}w_1 + \sum_{j=2}^{n} b_{j,k}w_j + \sum_{j=2}^{n} -b_{1,k}a_j w_j$$

$$= (b_{1,k} - b_{1,k})w_1 + \sum_{j=2}^{n}(b_{j,k}w_j - b_{1,k}a_j w_j)$$

$$= 0w_1 + \sum_{j=2}^{n}(b_{j,k} - b_{1,k}a_j)w_j.$$

Therefore, $A_{1,k} = 0$. Since ~~$A_{1,1}$ $A_{1,k} = 0$ we have the de~~ $A_{1,1} = 1$ and $A_{1,k} = 0$, we have the desired result.

Case 2: There's no $u \in V$ such that $Tu = 1w_1 + a_2w_2 + \cdots + a_nw_n$ for some scalars $a_2, \ldots, a_n$.

Let $v_1, \ldots, v_m$ be a basis of $V$ and $A = \mathcal{M}(T, (v_1, \ldots, v_m), (w_1, \ldots, w_n))$. Then for every $k \in \{1, \ldots, m\}$, $Tv_k = A_{1,k}w_1 + \cdots + A_{n,k}w_n$. If $A_{1,k} \neq 0$, then

$$T\left(\frac{1}{A_{1,k}}v_k\right) = \frac{1}{A_{1,k}}(Tv_k) = \frac{1}{A_{1,k}}\left(A_{1,k}w_1 + \cdots + A_{n,k}w_n\right) = \frac{A_{1,k}}{A_{1,k}}w_1 + \frac{A_{2,k}}{A_{1,k}}w_2 + \cdots + \frac{A_{n,k}}{A_{1,k}}w_n$$

$$= 1w_1 + \frac{A_{2,k}}{A_{1,k}}w_2 + \cdots + \frac{A_{n,k}}{A_{1,k}}w_n$$

which contradicts the Case 2 condition. Therefore, $A_{1,k} = 0$ which is our desired result.

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8. Suppose $A$ is an $m$-by-$n$ matrix and $B$ is an $n$-by-$p$ matrix. Prove that $(AB)_{j,\cdot} = A_{j,\cdot}B$ for each $1 \le j \le m$. In other words, show that row $j$ of $AB$ equals row $j$ of $A$ times $B$.

Because $A$ is a $m$-by-$n$ matrix and $B$ is an $n$-by-$p$ matrix, $AB$ is an $m$-by-$p$ matrix. Then $(AB)_{j,\cdot}$ is a 1-by-$p$ matrix. Since $A$ is a $m$-by-$n$ matrix, $A_{j,\cdot}$ is a 1-by-$n$ matrix, So $A_{j,\cdot}B$ is 1-by-$p$, the same size as $(AB)_{j,\cdot}$. ~~observe~~ For every $1 \le k \le p$ observe:
$((AB)_{j,\cdot})_{1,k} = (AB)_{j,k} = \sum_{r=1}^{n} A_{j,r}B_{r,k} = \sum_{r=1}^{n} (A_{j,\cdot})_{1,r}B_{r,k} = (A_{j,\cdot}B)_{1,k}$.

9. Suppose $a = (a_1 \cdots a_n)$ is a 1-by-$n$ matrix and $B$ is an $n$-by-$p$ matrix. Prove that $aB = a_1B_{1,\cdot} + \cdots + a_nB_{n,\cdot}$. In other words, show that $aB$ is a linear combination of the rows of $B$, with the scalars that multiply the rows coming from $a$.

Let $k$ be an arbitrary element of $\{1, \ldots, p\}$. Then observe:
$(aB)_k = \sum_{r=1}^{n} a_r B_{r,k} = a_1B_{1,k} + \cdots + a_nB_{n,k} = (a_1B_{1,\cdot} + \cdots + a_nB_{n,\cdot})_k$.

10. Give an example of 2-by-2 matricies $A$ and $B$ such that $AB \neq BA$.
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

~~11.~~ 12. Prove that matrix multiplication is associative. In other words, suppose $A$, $B$, and $C$ are ~~matrices~~ matrices whose size are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that $(AB)C = A(BC)$.

Let $m$ be the number of rows in $A$ and $n$ be the number of columns. Since $AB$ makes sense, $B$ must have $n$ rows. Let $p$ be the number of columns in $B$. Then $AB$ is an $m$-by-$p$ matrix. Since $(AB)C$ makes sense, $C$ must have $p$ rows. Let $q$ be the number of columns in $C$. ~~Then BC is an n~~ Because $B$ is $n$-by-$p$ and $C$ is $p$-by-$q$, $BC$ is valid and refers to an $n$-by-$q$ matrix. Therefore, $A(BC)$ is valid and refers to ~~an~~ an $m$-by-$q$ matrix.

For every $j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, q\}$, observe:
$$((AB)C)_{j,k} = \sum_{s=1}^{p} (AB)_{j,s}C_{s,k} = \sum_{s=1}^{p}\left(\sum_{r=1}^{n}(A_{j,r}B_{r,s})C_{s,k}\right) = \sum_{r=1}^{n}\left(A_{j,r}\sum_{s=1}^{p} B_{r,s}C_{s,k}\right)$$
$$= \sum_{r=1}^{n} A_{j,r}(BC)_{r,k} = (A(BC))_{j,k}.$$

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13. Suppose $A$ is an $n$-by-$n$ matrix and $1 \leq j,k \leq n$. Show that the entry in row $j$, column $k$ of $A^3$ is $\sum_{p=1}^{n}\sum_{r=1}^{n} A_{j,p}A_{p,r}A_{r,k}$

~~$A^3 \in (AA$ For every~~

$$A^3_{j,k} = ((AA)A)_{j,k} = \sum_{r=1}^{n}(AA)_{j,r}A_{r,k} = \sum_{r=1}^{n}\sum_{p=1}^{n}A_{j,p}A_{p,r}A_{r,k} = \sum_{p=1}^{n}\sum_{r=1}^{n}A_{j,p}A_{p,r}A_{r,k}$$

14. Suppose $m$ and $n$ are positive integers. Prove that the function $A \mapsto A^t$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$

Let $A$ and $B$ be $m$-by-$n$ matrices. Then observe how for every $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$, ~~$(A+B)^t$~~ $((A+B)^t)_{k,j} = (A+B)_{j,k} = A_{j,k} + B_{j,k} = (A^t)_{k,j} + (B^t)_{k,j} = (A^t+B^t)_{k,j}$. Let $\lambda \in \mathbf{F}$. Then $((\lambda A)^t)_{k,j} = (\lambda A)_{j,k} = \lambda A_{j,k} = \lambda (A^t)_{k,j} = (\lambda A^t)_{k,j}$. Therefore, $A \mapsto A^t$ is a linear map.

15. Prove that if $A$ is an $m$-by-$n$ matrix and $C$ is an $n$-by-$p$ matrix, then $(AC)^t = C^t A^t$.

For every $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,p\}$,
$((AC)^t)_{k,j} = (AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k} = \sum_{r=1}^{n}(A^t)_{r,j}(C^t)_{k,r} = \sum_{r=1}^{n}(C^t)_{k,r}(A^t)_{r,j} = (C^tA^t)_{k,j}$

16. Suppose $A$ is an $m$-by-$n$ matrix with $A \neq 0$. Prove that the rank of $A$ is 1 if and only if there exist $(c_1,\dots,c_m) \in \mathbf{F}^m$ and $(d_1,\dots,d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1,\dots,m$ and every $k = 1,\dots,n$.

Assume that rank $A = 1$, ~~the there's~~ then there's some 1-by-$n$ matrix $D = (d_1 \cdots d_n)$ such that $D$ is a basis of the span of $A$'s rows. Let $j$ be an arbitrary element of $\{1,\dots,m\}$ and $k$ be an arbitrary element of $\{1,\dots,n\}$. Then for every $j$, there must be some $c_j \in \mathbf{F}$ such that $A_{j,\cdot} = c_j D$.

~~So for every $k$ $A$~~ Now observe:
$A_{j,k} = (A_{j,\cdot})_k = (c_j D)_k = c_j D_k = c_j d_k$. Then $(c_1,\dots,c_m)$ and $(d_1,\dots,d_n)$ are the desired vectors in $\mathbf{F}^m$ and $\mathbf{F}^n$.

For the converse assume that there's some $(c_1,\dots,c_m) \in \mathbf{F}^m$ and some $(d_1,\dots,d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$. Then span$(A_{1,\cdot},\dots,A_{m,\cdot}) =$
$\{a_1 A_{1,\cdot} + \cdots + a_m A_{m,\cdot} : a_1,\dots,a_m \in \mathbf{F}\} = \{a_1(A_{1,1} \cdots A_{1,n}) + \cdots + a_m(A_{m,1} \cdots A_{m,n}) : a_1,\dots,a_m \in \mathbf{F}\} =$
$\{a_1(c_1d_1 \cdots c_1d_n) + \cdots + a_m(c_md_1 \cdots c_md_n) : a_1,\dots,a_m \in \mathbf{F}\} = \{a_1c_1(d_1 \cdots d_n) + \cdots + a_mc_m(d_1 \cdots d_n) : a_1,\dots,a_m \in \mathbf{F}\}$.

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So $\operatorname{span}(A_{1,\cdot},\dots,A_{m,\cdot})=\{(a_1c_1+\cdots+a_mc_m)(d_1\cdots d_n):a_1,\dots,a_m\in\mathbf{F}\}=\{a(d_1\cdots d_n):a\in\mathbf{F}\}=\operatorname{span}((d_1\cdots d_n))$.

If $(d_1\cdots d_n)=0$, then $\operatorname{span}((d_1\cdots d_n))=\{0\}$. So for every $j=1,\dots,m$ and $k=1,\dots,n$ $A_{j,k}=(A_{j,\cdot})_k$. Since $A_{j,\cdot}\in\operatorname{span}((d_1\cdots d_n))=\{0\}$, $A_{j,\cdot}=0$ so $A_{j,k}=(A_{j,\cdot})_k=0_k=0$ which implies $A=0$. However, because $A\neq0$, we know that $(d_1\cdots d_n)\neq0$.

Then $(d_1\cdots d_n)$ is a linearly independent list that spans $\operatorname{span}(A_{1,\cdot},\dots,A_{m,\cdot})$ so $(d_1\cdots d_n)$ is a basis of $\operatorname{span}(A_{1,\cdot},\dots,A_{m,\cdot})$. Therefore, $\dim\operatorname{span}(A_{1,\cdot},\dots,A_{m,\cdot})=1$ which implies the rank of $A$ is 1.

17. Suppose $T\in\mathcal{L}(V)$ and $u_1,\dots,u_n$ and $v_1,\dots,v_n$ are bases of $V$. Prove that the following are equivalent.
(a) $T$ is injective
(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$
(c) The columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$
(d) The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$
(e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$

First we show that (a) iff (b). Fix $A=\mathcal{M}(T)$ for this entire proof.

Assume (a) $T$ is injective. Then we can show that $Tu_1,\dots,Tu_n$ is linearly independent by considering scalars $a_1,\dots,a_n$ such that $0=a_1Tu_1+\cdots+a_nTu_n=T(a_1u_1+\cdots+a_nu_n)$. Then $a_1u_1+\cdots+a_nu_n\in\operatorname{null}T=\{0\}$. So $a_1u_1+\cdots+a_nu_n=0$ which implies $a_1=\cdots=a_n=0$.

Now we show (b) the columns of $\mathcal{M}(T)$ are linearly independent by considering scalars $a_1,\dots,a_n$ such that

$$0=\sum_{k=1}^{n}a_kA_{\cdot,k}=\sum_{k=1}^{n}a_k\begin{pmatrix}A_{1,k}\\ \vdots\\ A_{n,k}\end{pmatrix}=\sum_{k=1}^{n}\begin{pmatrix}a_kA_{1,k}\\ \vdots\\ a_kA_{n,k}\end{pmatrix}=\begin{pmatrix}\sum_{k=1}^{n}a_kA_{1,k}\\ \vdots\\ \sum_{k=1}^{n}a_kA_{n,k}\end{pmatrix}$$

So for every $j=1,\dots,n$ $0=\sum_{k=1}^{n}a_kA_{j,k}$. Now observe

$$\sum_{k=1}^{n}a_kTu_k=\sum_{k=1}^{n}a_k\Big(\sum_{j=1}^{n}A_{j,k}v_j\Big)=\sum_{j=1}^{n}\sum_{k=1}^{n}a_kA_{j,k}v_j=\sum_{j=1}^{n}\Big(\sum_{k=1}^{n}a_kA_{j,k}\Big)v_j=\sum_{j=1}^{n}0v_j=0.$$

Since $Tu_1,\dots,Tu_n$ is linearly independent, this implies that $a_1=\cdots=a_n=0$.

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For the converse assume (b) the columns of $A$ are linearly independent. To show that $T$ is injective, we show that null $T = \{0\}$. Let $w \in$ null $T$. Since $u_1, \dots, u_n$ is a basis of $V$, there's some scalars $a_1, \dots, a_m$ such that $w = a_1 u_1 + \dots + a_n u_n$. Because $w \in$ null $T$,

$$0 = Tw = \sum_{k=1}^{n} a_k T u_k = \sum_{k=1}^{n} a_k \left( \sum_{j=1}^{n} A_{j,k} v_j \right) = \sum_{j=1}^{n} \left( \sum_{k=1}^{n} a_k A_{j,k} \right) v_j$$

which implies that for every $j = 1, \dots, n$ $\sum_{k=1}^{n} a_k A_{j,k} = 0$. Now observe

$$\sum_{k=1}^{n} a_k A_{\cdot,k} = \sum_{k=1}^{n} a_k \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{n,k} \end{pmatrix} = \sum_{k=1}^{n} \begin{pmatrix} a_k A_{1,k} \\ \vdots \\ a_k A_{n,k} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} a_k A_{1,k} \\ \vdots \\ \sum_{k=1}^{n} a_k A_{n,k} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

Because the columns of $A$ are linearly independent, this implies that $a_1 = \dots = a_n = 0$. So $w = a_1 u_1 + \dots + a_n u_n = 0u_1 + \dots + 0u_n = 0$.

Next we show that (b) iff (c).

Assume (b) the columns of $A$ are linearly independent. ~~Notice how the list of n n-by-1 matrices where the kth matrix has a kth entry of 1 and all other entries set to 0 is a basis of $\mathbf{F}^{n,1}$. Then dim $\mathbf{F}^{n,1} = n$.~~
By theorem 3.40 dim $\mathbf{F}^{n,1} = n1 = n$. Then since the columns of $A$ are a linearly independent list of length $n = \dim \mathbf{F}^{n,1}$, we know by theorem 2.38 that they're a basis of $\mathbf{F}^{n,1}$ which implies that they span $\mathbf{F}^{n,1}$ (c).

Assume (c) the columns of $A$ span $\mathbf{F}^{n,1}$. Then since the columns of $A$ are a length $n = \dim \mathbf{F}^{n,1}$ that spans $\mathbf{F}^{n,1}$, theorem 2.42 tells us they're a basis of $\mathbf{F}^{n,1}$, which implies that they're linearly independent in $\mathbf{F}^{n,1}$ (b).

By similar logic, (d) iff (e).

Now we show that (c) iff (d).

Assume (c) the columns of $A$ span $\mathbf{F}^{n,1}$. Then the rank of $A$ is $n$. So the dimension of the ~~rows of A is n~~ span of the rows of $A$ is $n$. Because a subspace of full dimension equals the whole space (2.39), we know that (d) the rows of $A$ span $\mathbf{F}^{1,n}$.

Assume (d) the rows of $A$ span $\mathbf{F}^{1,n}$. Then the rank of $A$ is $n$. Then the dimension of the span of the columns of $A$ is $n$. So by theorem 2.39, (c) the columns of $A$ span $\mathbf{F}^{n,1}$. Then (a) iff (b) iff (c) iff (d) iff (e).